

Putnam Club Problem Sheet - February 14

Warm-up

Putnam 2006 A1. Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

Putnam 2010 A3. Suppose that the function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a \frac{\partial h}{\partial x}(x, y) + b \frac{\partial h}{\partial y}(x, y)$$

for some constants a, b . Prove that if there is a constant M such that $|h(x, y)| \leq M$ for all $(x, y) \in \mathbb{R}^2$, then h is identically zero.

Putnam 2004 B3. Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y); 0 \leq x \leq a, 0 \leq y \leq f(x)\}$$

has perimeter k units and area k square units for some real number k .

Harder Problems

Putnam 2011 A5. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be twice continuously differentiable functions with the following properties:

- $F(u, u) = 0$ for every $u \in \mathbb{R}$;
- for every $x \in \mathbb{R}$, $g(x) > 0$ and $x^2 g(x) \leq 1$;
- for every $(u, v) \in \mathbb{R}^2$, the vector $\nabla F(u, v)$ is either $\mathbf{0}$ or parallel to the vector $\langle g(u), -g(v) \rangle$.

Prove that there exists a constant C such that for every $n \geq 2$ and any $x_1, \dots, x_{n+1} \in \mathbb{R}$, we have

$$\min_{i \neq j} |F(x_i, x_j)| \leq \frac{C}{n}.$$

Putnam 2004 A6. Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x, y) dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x, y) dy \right)^2 dx \leq \left(\int_0^1 \int_0^1 f(x, y) dx dy \right)^2 + \int_0^1 \int_0^1 [f(x, y)]^2 dx dy.$$

Putnam 2009 A6. Let $f : [0, 1]^2 \rightarrow \mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0, 1)^2$. Let $a = \int_0^1 f(0, y) dy$, $b = \int_0^1 f(1, y) dy$, $c = \int_0^1 f(x, 0) dx$, $d = \int_0^1 f(x, 1) dx$. Prove or disprove: There must be a point (x_0, y_0) in $(0, 1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0, y_0) = b - a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = d - c.$$

Putnam 2018 B5. Let $f = (f_1, f_2)$ be a function from \mathbb{R}^2 to \mathbb{R}^2 with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} - \frac{1}{4} \left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} \right)^2 > 0$$

everywhere. Prove that f is one-to-one.

Two Devilish Problems

Putnam 2012 A6. Let $f(x, y)$ be a continuous, real-valued function on \mathbb{R}^2 . Suppose that, for every rectangular region R of area 1, the double integral of $f(x, y)$ over R equals 0. Must $f(x, y)$ be identically 0?

Putnam 2003 B6. Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \geq \int_0^1 |f(x)| \, dx.$$