Putnam Club. Spring 2022. Probability (Apr 20).

1. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

2. Peter tosses 25 fair coins and John tosses 20 fair coins. What is the probability that they get the same number of heads?

3. (a) Choose 3 vertices of regular $(2n+1)$-polygon at random. What is the probability that the resulting triangular is acute?

    (b) Choose 3 points on a circle at random. What is the probability that the resulting triangular is acute?

4. You have a set of n biased coins. The mth coin has probability $1/(2m+1)$ of landing heads $(m = 1, 2, \ldots, n)$ and the results for each coin are independent. What is the probability that if each coin is tossed once, you get an odd number of heads?

5. Find the number of subsets of $\{1, 2; \ldots, n\}$ that contain no two consecutive elements.

6. We pick n points at random on a circle. What is the probability that the center of the circle will be in the convex polygon with vertices at those points?

7. Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing the number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. If we assume optimal strategy by both A and B, what is the probability that A wins?

8. Let $S = \{1, 2 \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

$$
\begin{array}{ccc}
\text{(i)} & \pi(k) > \pi(k+1) & \text{for } k = 1 \\
\text{(ii)} & \pi(k-1) < \pi(k) \text{ and } \pi(k) > \pi(k+1) & \text{for } 1 < k < n \\
\text{(iii)} & \pi(k-1) < \pi(k) & \text{for } k = n
\end{array}
$$

(For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 as $k = 1$, and a local maximum at $k = 4$.) What is the average number of local maxima of a permutation of S, averaging (uniformly) over all permutations of S?

9. On the unit circle centered at the origin $(x^2 + y^2 = 1)$ we pick three points at random. We cut the circle into three arcs at those points. What is the expected length of the arc containing the point $(1; 0)$?

10. Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k + 1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3 ? Your answer should be in closed form, but may include factorials.

11. Suppose X is a random variable that takes on only non-negative integer values, with $E[X] = 1$, $E[X^2] = 2$, and $E[X^3] = 5$. (Here $E[Y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X = 0$.

Hints.

1. Prove that if she attempts n shots, the probability she hit any number of shots from 1 to $n-1$ is equal.

2. The probability of John getting n heads is the same as that of he getting n tails. How many heads will they get together?

3. Compute this probability if one vertex is fixed.

4. Consider the expression $(2/3 - 1/3)(4/5 - 1/5)(6/7 - 1/7)\ldots(2n/(2n+1) - 1/(2n+1))$ and think what it means probabilistically.

5. Find a bijection between the k-element subsets of $\{1, 2, \ldots, n\}$ with no consecutive elements and all k-element subsets of $\{1, 2, \ldots, n-k+1\}$.

6. Find the probability of the polygon NOT containing the center of the circle.

7. The stuff about random shuffling and probabilities is a red herring: B always wins

8. One way is to count number of times each position is max.

9. The problem is equivalent to dividing a circle of length 2π at four points chosen at random, and label one of them $(1; 0)$.

10. Think of all of the numbers as being $1, 0$, or -1 (i.e. consider mod 3). We have $k+1$ copies of 1 and k copies each of 0 and -1. The final sum is going to be 1. To get there, not changing the sum by more than one each time, and avoiding multiples of 3, then all intermediate sums must be 1 or 2.

11. work with the series for moments