Putnam Club. Spring 2022. Analysis (Feb 16, Feb 23).

Easier problems

1. Find the value of the following infinitely nested radical

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$$

2. Prove that for $n \geq 2$, the equation $x^n + x - 1 = 0$ has a unique root in the interval $[0, 1]$. If x_n denotes this root, prove that the sequence $\{x_n\}_{n=2}^{\infty}$ is convergent and find its limit.

3. Compute $\lim\limits_{n\to\infty} \left\{ \prod\limits_{k=1}^{n} \left(1 + \frac{k}{n}\right) \right\}^{1/n}$.

4. Let $f : [0; 1] \to \mathbb{R}$ continuous, and suppose that $f(0) = f(1)$. Show that there is a value $x \in [0; 1998/1999]$ satisfying $f(x) = f(x + 1/1999)$.

5. Prove that there are no positive numbers x and y such that $x2^y + y2^{-x} = x + y$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $f(x) \cdot f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.

Putnam problems

7. (Putnam 1995) Evaluate

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \ldots}}}$$

Express your answer in the form $(a + b\sqrt{c})/d$, where a, b, c, d, are integers.

8. (Putnam 1992) Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f(\tfrac{1}{n}) = \frac{n^2}{n^2+1}, \quad n = 1, 2, 3, \ldots$$

compute the values of the derivatives $f^{(k)}(0)$, $n = 1, 2, 3, \ldots$.

9. (Putnam 2016) Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

10. Let α be a real number such that n^α is an integer for every positive integer n. Prove that α is a non-negative integer.

11. (Putnam 2014) Suppose that f is a function on the interval $[1, 3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?

12. (Putnam 2016) Suppose that f is a function from $\mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$$

for all real $x \neq 0$. (As usual, $y = \arctan x$ means $-\pi/2 < y < \pi/2$ and $\tan y = x$.) Find

$$\int_0^1 f(x)\, dx.$$

13. (Putnam 2017) Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$ but $f \neq g$. For every positive integer n, define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\, dx.$$

Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.