Pick's Theorem

Eva Elduque

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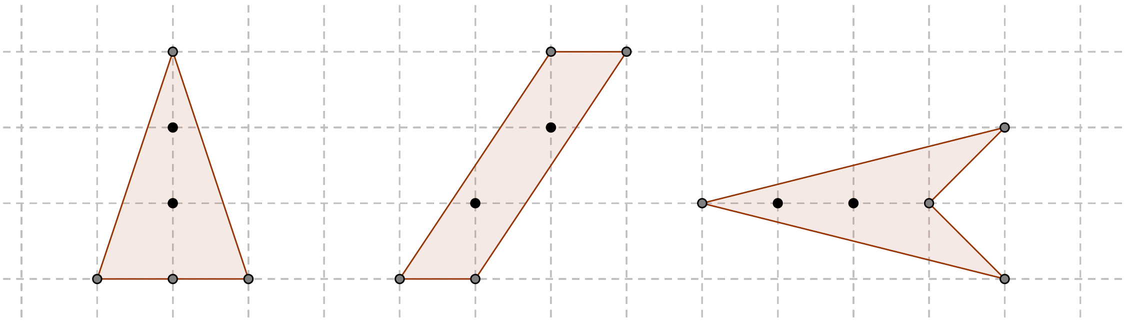
ANSWER KEY

Today, we will talk about one surprising result called Pick's Theorem, which will allow us to easily compute the area of a simple polygon (a polygon with no holes) whose vertices are on points of an evenly spaced grid. To compute the area of a simple polygon this way, we will only need to know the number of **interior points** (the points of the grid that are inside the polygon) and the number of **boundary points** (the points of the grid that lie exactly in the boundary of the polygon). In this talk, we will only speak about simple polygons whose vertices lie on points of an evenly spaced grid, so from now on we will call them just polygons. An area unit will be the area of one of the squares forming the grid.

First of all, let's see if we believe that a result like that can exist. If the only things that we have to take into account to compute the area of a polygon are its interior points and its boundary points, two polygons with the same number of interior and boundary points should have the same area.

Question 1: Draw several polygons in a grid with the same number of interior and boundary points, and compute their area. Are those areas the same?

Answer. One possible example is these three polygons:



The three polygons have 2 interior points and 4 boundary points, and all of them have area 3. □

Now we have a reason to believe that a formula to compute the area based only on the number of boundary and interior points exists, so let's try to find it. From now on, we will use the letters $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{I}$ to denote area, number of boundary points and number of interior points of a polygon respectively. Thus, what we want to find is a formula of the form

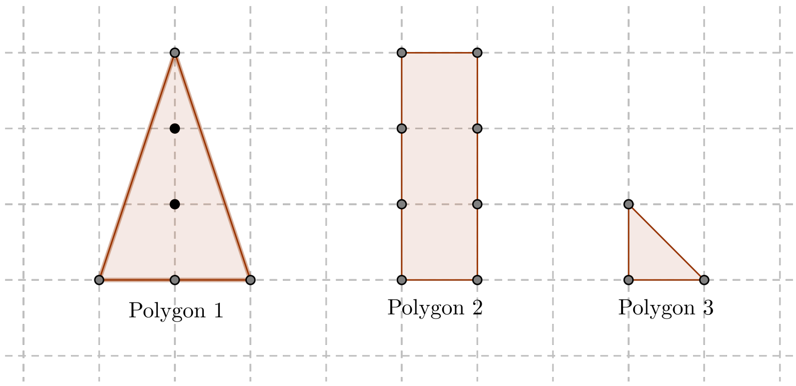
$$\mathcal{A} = \text{ something that depends only on } \mathcal{I} \text{ and } \mathcal{B}$$

Question 2: If you knew that the formula we are looking for was of the form

$$\mathcal{A} = a\mathcal{I} + b\mathcal{B} + c$$

for some numbers a, b and c, what would those numbers a, b and c be?

Answer. The way to approach this question is to do some examples first. We need to start by drawing some polygons whose areas we know how to compute already. For example, I drew



and I collected the following data:

	Polygon 1	**Polygon 2**	**Polygon 3**
$\mathcal{I}$	2	0	0
$\mathcal{B}$	4	8	3
$\mathcal{A}$	3	3	$\frac{1}{2}$

Pluging that numbers into the formula $\mathcal{A} = a\mathcal{I} + b\mathcal{B} + c$ we get the following system of equations

$$\begin{cases} 3 = 2a + 4b + c \\ 3 = 8b + c \\ \frac{1}{2} = 3b + c \end{cases}$$

If we solve the last two equations, we get

$$b = \frac{1}{2}c = -1$$

and putting those values of b and c in the first equation we get that

$$a = 1$$

so, if the formula we are looking for has the form

$$\mathcal{A} = a\mathcal{I} + b\mathcal{B} + c$$

then, it would have to be

$$\mathcal{A} = \mathcal{I} + \frac{1}{2}\mathcal{B} - 1$$

□

If we keep testing the formula we just got ($\mathcal{A} = \mathcal{I} + \frac{1}{2}\mathcal{B} - 1$) with other polygons, we will see that it also works for those! In fact, that formula is exactly Pick's Theorem. Let's write it down nicely:

Pick's Theorem. *The area of every simple polygon whose vertices lie on points of a grid is*

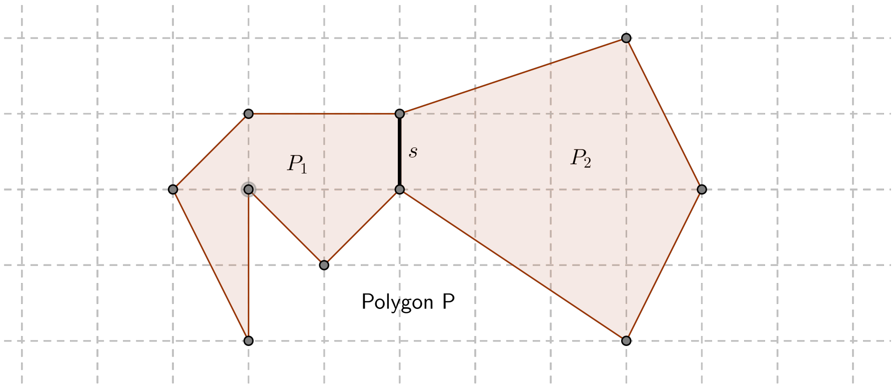
$$\mathcal{A} = \mathcal{I} + \frac{1}{2}\mathcal{B} - 1$$

Now it is our turn to prove it, and we will do it following some guidelines. Polygons can be as complicated as we want to make them, so it would be nice if we could break up a polygon into smaller ones that were easier. To use this in our proof, we need to answer the following question:

Question 3: We denote by $\mathcal{F}$ the right hand side in the formula in Pick's Theorem, that is

$$\mathcal{F} = \mathcal{I} + \frac{1}{2}\mathcal{B} - 1$$

Let P_1 and P_2 be two polygons joined by a common side s, and let P be the big polygon that is the union of P_1 and P_2. Prove that $\mathcal{F}_P = \mathcal{F}_{P_1} + \mathcal{F}_{P_2}$.



Answer. The interior points of the polygon P are the union of the interior points of P_1, the interior points of P_2 and the points of the grid that lie in the interior of the common side s. We write this fact like this:

$$\mathcal{I}_P = \mathcal{I}_{P_1} + \mathcal{I}_{P_2} + \mathcal{I}_s$$

Now, let's count the boundary points of the polygon P. If we sum $\mathcal{B}_{P_1}$ and $\mathcal{B}_{P_2}$, we have counted the two points in the boundary of s twice (and we only want to count those once), and the points of the interior of s twice as well (and we don't want to count them at all, since they are not in the boundary of the polygon P). Hence, we should substract the things we've counted more times than we wanted to come up with the correct answer, like so:

$$\mathcal{B}_P = \mathcal{B}_{P_1} + \mathcal{B}_{P_2} - 2\mathcal{I}_s - 2$$

Using up these two formulas that we just wrote down, we get

$$\mathcal{F}_P = (\mathcal{I}_{P_1} + \mathcal{I}_{P_2} + \mathcal{I}_s) + \frac{1}{2}(\mathcal{B}_{P_1} + \mathcal{B}_{P_2} - 2\mathcal{I}_s - 2) - 1 =$$

$$= (\mathcal{I}_{P_1} + \frac{1}{2}\mathcal{B}_{P_1} - 1) + (\mathcal{I}_{P_2} + \frac{1}{2}\mathcal{B}_{P_2} - 1) =$$

$$= \mathcal{F}_{P_1} + \mathcal{F}_{P_2}$$

□

Using Question 3, the following is an easier exercise.

Question 4: Suppose Pick's Theorem holds for a polygon P_1 and for a polygon P_2 joined by a side s like the ones in Question 3, that is

$$\mathcal{A}_{P_1} = \mathcal{F}_{P_1}$$

and

$$\mathcal{A}_{P_2} = \mathcal{F}_{P_2}$$

Prove that, in this case, Pick's Theorem also holds for the polygon P, that is, prove that $\mathcal{A}_P = \mathcal{F}_P$.

Answer. The area of P is the sum of the areas of P_1 and P_2, that is

$$\mathcal{A}_P = \mathcal{A}_{P_1} + \mathcal{A}_{P_2}$$

By Question 3,

$$\mathcal{F}_P = \mathcal{F}_{P_1} + \mathcal{F}_{P_2}$$

Since, in this question we have that $\mathcal{A}_{P_1} = \mathcal{F}_{P_1}$ and $\mathcal{A}_{P_2} = \mathcal{F}_{P_2}$, we get that

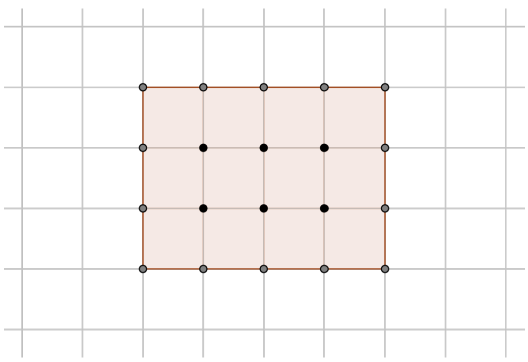
$$\mathcal{A}_P = \mathcal{F}_P$$

□

The answer to this last question will be very important in our proof. Imagine that we have a certain class of nice polygons that satisfy that you can construct any other polygon by joining nice polygons by common sides. Imagine also that the nice polygons satisfy Pick's Theorem. Then, by Question 4, the theorem will be proved! That is, every polygon will satisfy Pick's Theorem. But every polygon can be decomposed into triangles, so we just have to prove that Pick's Theorem is true for triangles, and that will show (by Question 4) that it is true for **all** the polygons!

Math sometimes works in weird ways: sometimes it's very hard to prove certain things directly, but it's easier if you have the help of another result that's easier to prove but might not seem related to what we want to prove in the first place. An example of this is what we are going to do now. To see that Pick's Theorem is true for triangles, we are going to use that it is true for rectangles with horizontal and vertical sides.

Question 5: Prove that Pick's Theorem holds for rectangles that have horizontal and vertical sides, such as the one drawn in the picture.



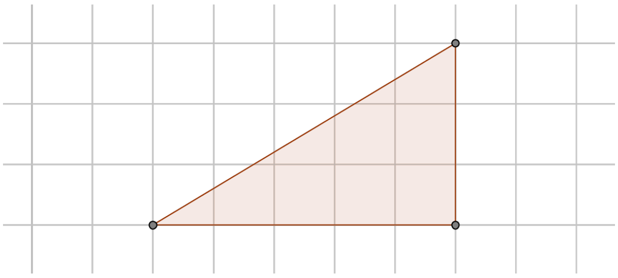
Answer. Let R be one such rectangle. It can be seen as the union of little squares of side 1, the squares the grid forms inside the rectangle. By the answer of Question 4, we just have to prove it for the little squares the grid forms. Those have area 1, 4 boundary points and no interior points. Those values satisfy Pick's Theorem, since

$$1 = 0 + \frac{1}{2}4 - 1$$

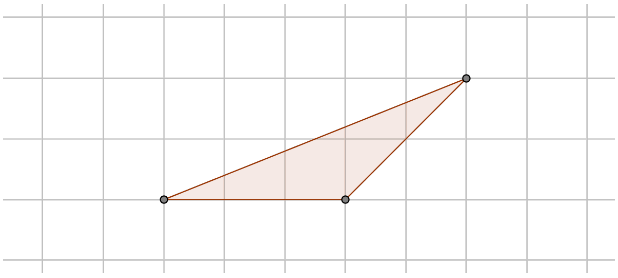
□

Now, we want to use Question 5 to prove Pick's Theorem for triangles, the last step we have left. To do that, we classify triangles on a grid into 3 groups, namely

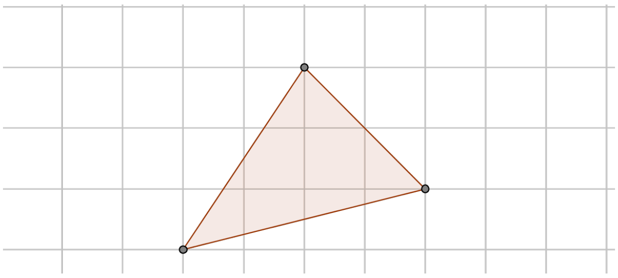
1. Right triangles that have one horizontal and one vertical side.



2. Triangles that only have one horizontal (or vertical) side.

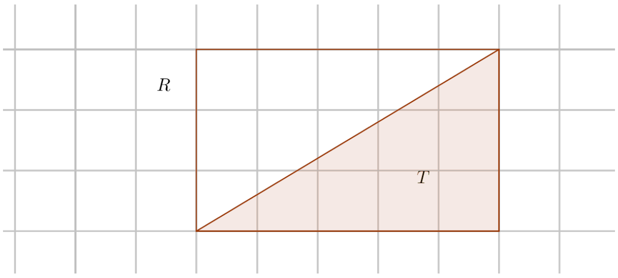


3. Triangles that don't have any vertical or horizontal sides.



To prove Pick's Theorem for any of these cases, we will use our previous work by inscribing our triangle in a rectangle like those in Question 5.

Question 6: Prove Pick's Theorem for the triangles T of type 1 (Right triangles that have one horizontal and one vertical side). To do this, use the following picture, which represents the triangle T inscribed in a rectangle R with horizontal and vertical sides



Answer. The rectangle R is the union of the triangle T and another triangle that's exactly like T. By Question 3,

$$\mathcal{F}_R = 2\mathcal{F}_T$$

Also, the area of R is two times the area of T, that is,

$$\mathcal{A}_R = 2\mathcal{A}_T$$

By Question 5, Pick's Theorem holds for R, that is

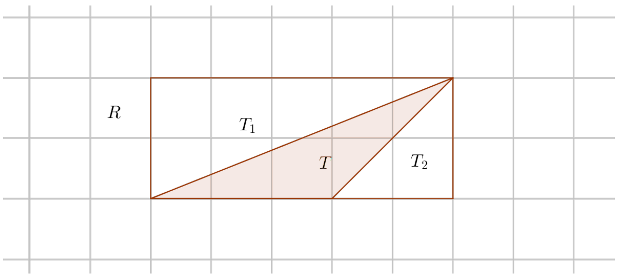
$$\mathcal{A}_R = \mathcal{F}_R$$

Hence, substituting A_R and $\mathcal{F}_R$ in that last equation, and dividing everything by 2, we get

$$\mathcal{A}_T = \mathcal{F}_T$$

and Pick's Theorem holds for the triangle T, like we wanted to prove. □

Question 7: Prove Pick's Theorem for the triangles T of type 2 (Triangles that only have one horizontal (or vertical) side). To do this, use the following picture, which represents the triangle T inscribed in a rectangle R with horizontal and vertical sides.



Answer. The rectangle R is the union of the triangles T, T_1 and T_2, where T_1 and T_2 are triangles of type 1. By Question 3,

$$\mathcal{F}_R = \mathcal{F}_T + \mathcal{F}_{T_1} + \mathcal{F}_{T_2}$$

Also, by Question 6

$$\mathcal{A}_{T_1} = \mathcal{F}_{T_1}$$
$$\mathcal{A}_{T_2} = \mathcal{F}_{T_2}$$

and by Question 5

$$\mathcal{A}_R = \mathcal{F}_R$$

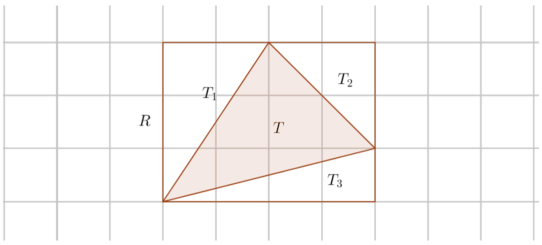
Putting together the last four equalities we get that

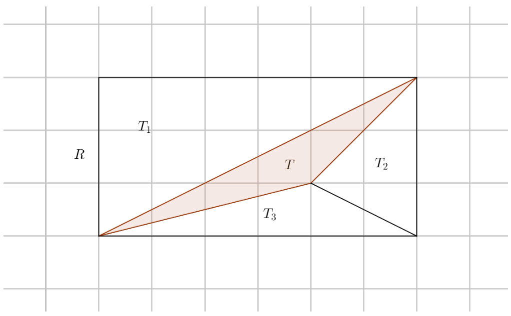
$$\mathcal{A}_T = \mathcal{F}_T$$

and Pick's Theorem holds for the triangle T, like we wanted to prove. □

Finally, to complete the proof of Pick's Theorem, all that's left to prove is

Question 8: Prove Pick's Theorem for the triangles T of type 3 (Triangles that don't have any vertical or horizontal sides). To do this, use the following pictures, which represent the triangle T inscribed in a rectangle R with horizontal and vertical sides. In the first picture all three vertices of T lie in the boundary of R, and in the second, only two of them do.





Answer. The rectangle R is the union of the triangles T, T_1, T_2 and T_3, where T_1, T_2 and T_3 are triangles of type 1 or 2. Since in Questions 5, 6 and 7 we've proved that Pick's Theorem holds for R, T_1 and T_2 and T_3 respectively, arguing like we did in the answer of the answer of Questions 6 and 7 we get that Pick's Theorem must hold for T too. □

We are done! Now, we know for sure that Pick's Theorem is true, because we just proved it, so we can use the formula to calculate areas from now on with no worries.

Apart from computing areas, Pick's Theorem has more applications. Here, we are just going to give one example of this, in the form of an extension problem for those students who wish to do it.

Extension Problem: Using Pick's theorem, prove that it's impossible to draw an equilateral triangle whose vertices are on a grid.

Hint: For those who don't know trigonometry, if b and h are the base and height of an equilateral triangle, then $h = \frac{\sqrt{3}}{2}b$. Use also that $\sqrt{3}$ is an irrational number, that is, it can't be written as a fraction $\frac{a}{b}$, where a and b are integers.

Answer. Suppose that we have an equilateral triangle T such that its vertices are on a grid, and let b be its base, and h its height. Note that the segment b has its endpoints on points of the grid.

The area of out equilateral triangle is:

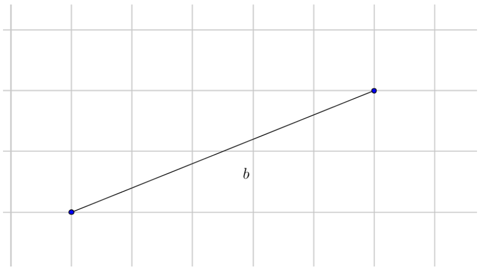
$$\mathcal{A}_T = \frac{bh}{2} = \frac{\sqrt{3}b^2}{4}$$

by Pick's theorem, we also know that

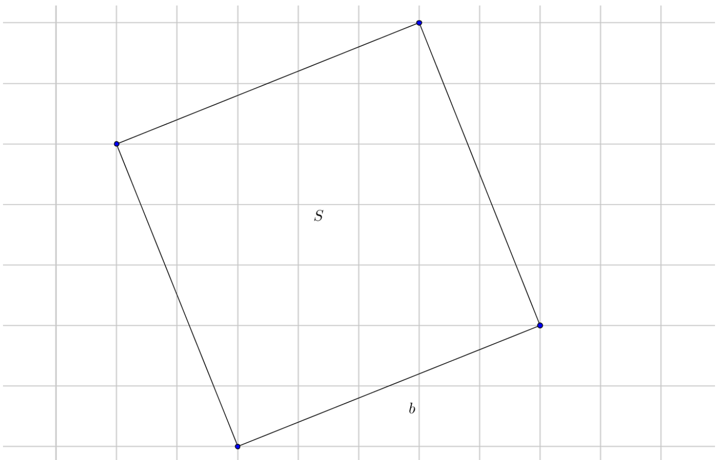
$$\mathcal{A}_T = \mathcal{I}_T + \frac{1}{2}\mathcal{B}_T - 1$$

so in particular, $\mathcal{A}_T = \frac{\sqrt{3}b^2}{4}$ is a rational number, that is, it can be written as a fraction $\frac{x}{y}$ where x and y are integers, and y is not 0.

The segment b has endpoints on points in the grid, like so



That means that we can draw a square S with vertices on the grid from our segment b like so



This square has area b^2. By Pick's Theorem,

$$\mathcal{A}_S = \mathcal{I}_S + \frac{1}{2}\mathcal{B}_S - 1$$

so in particular, b^2 is a rational number.

We have that both b^2 and $\frac{\sqrt{3}b^2}{4}$ are positive (they correspond to areas) rational numbers. From this it follows that $\sqrt{3}$ is a rational number, because

$$\sqrt{3} = 4\frac{\left(\frac{\sqrt{3}b^2}{4}\right)}{b^2}$$

and multiplying and dividing rational numbers gives back another rational number. But $\sqrt{3}$ is **not** a rational number (look at the hint)! We just reached a contradiction, that comes from assuming that an equilateral triangle T whose vertices are on points of a grid exists. Hence, no such equilateral triangle can exist. □

References

[1] Elduque, Alberto. `http://www.unizar.es/ttm/2010-11/Pick11.pdf`
A lecture about Pick's Theorem from an activity similar to Math Circle that takes place in the University of Zaragoza, Spain (in Spanish).

[2] Yaggie, Jon. `http://math.sfsu.edu/cm2/papers/jon.pdf`
Another Math Circle talk about Pick's Theorem.