Fall 2025 Seminar: Geometric Satake Correspondence

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1 The Geometic Satake Correspondence

Let k be a field of characteristic zero. Let G be a complex connected reductive group. Then the Geometric Satake Correspondence (GSC) asserts that there is an equivalence of *tensor* categories between $\mathrm{Rep}(G^\vee, k)$ and $\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G, k)$. The former category is the category of finite dimensional k-representations of $G^\vee$ which is the split reductive group over k whose root datum is dual to G (aka $G^\vee$ is the Langlands dual group to G). The latter category is the category of $G_\mathcal{O}$-invariant perverse sheaves over k. It is sometimes referred to as the Satake Category. The $G_\mathcal{O}$-orbits induce a stratification of Gr_G and so under this setting, we are consider perverse sheaves with respect to the induced stratification $\mathcal{S}$. To describe the equivalence of tensor categories of above, we will need to provide $\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G, k)$ with a *convolution product*.

The proof of GSC by Mirković and Vilonen [6] utilizes the Tannakian formalism as developed by Deligne and Milne. The main tool is Theorem 2.11 of [3] which asserts a neutral Tannakian category is equivalent to the category of finite dimensional k-representations over some group scheme $\widetilde{G}$. So in the case of the GSC, upon checking the axioms of a neutral Tannakian category, it remains then to determine which group scheme $\widetilde{G}$ should be. One must do some work here e.g. checking that $\widetilde{G}$ is an algebraic group which is split connected and reductive and whose root datum is that of $G^\vee$. It is worth emphasizing that this identifying $\widetilde{G}$ with $G^\vee$ is the substantial step.

The history of the Geometric Satake Correspondence goes back to work by Lutszig but the first proof of the theorem in the case of $k = \mathbb{C}$ was given by V. Ginzburg. Then, [6] provides a proof for general coefficients. There have been many recent works on GSC by others, but for the purposes of this seminar, we will focus on the approach used by [6] and work with $k = \mathbb{C}$. The proof in the case of general coefficients k requires more work and [2] expand on the details in [6]. If there is some time, we can discuss how to obtain GSC for general coefficients.

2 Outline of Talks

This is an outline for a seminar on the Geometric Satake Correspondence. It will be divided up into an undetermined number talks. Edits and adjustments will be made over time. We will aim to have talks occur weekly / biweekly depending on the week involved. Volunteers should expect to be ready to speak on the day they have volunteered for.

Disclosure on choice of topics: Some of the topics chosen are not all immediately related to the proof of GSC. Indeed, there are some topics which are tangential but might be useful / encouraging to learn before delving into the proof.

For resources in preparing for talks, please see the appropriate sections below.

§1: Semisimple Lie Algebras over $\mathbb{C}$

Content: Summarize the classification of semisimple Lie algebras over $\mathbb{C}$, describe the representation theory and recall the correspondence between irreducible representations and integral dominant weights.

References: [4]

§2: Reductive Algebraic Groups and Root Datum

*This is a draft outline. Typos and mistakes are expected. There are also some inaccuracies in this outline, but because I am lazy, I won't fix them.

Content: Define reductive algebraic groups, explain terminology related to it e.g. (a) what is a connected algebraic group?, (b) what is a split reductive group? or (c) any other definitions needed to introduce root datum. Define root datum for reductive connected algebraic groups. Show how to take an algebraic group and a split maximal torus (G, T) and obtain root datum. Define Borel subgroup. Explain how to do representation theory for reductive connected algebraic groups: there is a bijection between irreducible representations of G, dominant weights in $X^*(T)$, and irreducible representations of $\mathfrak{g}$, and etc. Explain that $\mathrm{Rep}_k(G)$ is semisimple so this classification clarifies everything needed.

If there is time, discuss the Weyl group and Bruhat group. Also, give examples of Langlands dual groups.

References: [5]. Also consider looking at the Lecture 2 Notes from a class by D. Nadler for ideas.

§3: Line Bundles on Flag Varieties and Beilinson Bernstein Localization

Content: While doing everything here, it is best to have on the side the details for when $G := \mathrm{SL}_n$ and $B := \{\text{upper triangular matrices}\}$. Then $G/B = \mathbb{P}^1_{\mathbb{C}}$. Define flag variety G/B for a given complex algebraic group G. Describe line bundles on G/B and the correspondence with irreducible representations. Move towards the statement of Beilinson-Bernstein localization. Summarize necessary definitions and work out BB localization for $\mathbb{P}^1$ (one can be very explicit here e.g. A. Romanov's talk). For the statement of BB localization and some consequences, see [8].

References: [8]

§4: The Tannakian Reconstruction Theorem

Content: This is an "easier" talk compared to the previous one. Work towards stating Theorem 2.11 of [3]. Give the necessary definitions. One should look at [2] for expanded details. Since we not working over general coefficients at this point, you might as well work over $\mathbb{C}$. Do not try to prove the theorem. Following [2] closely is likely the move since one can have a very short talk if one only states the theorem, but really, you should try to describe properties of G visible from $\mathrm{Rep}(G)$.

References: [2], [3]

§5: Introduction to Perverse Sheaves

Content: This talk is meant for people who do not know perverse sheaves that well or have seen the theory too much and need a refresher. Define perverse sheaves, recall the Riemann-Hilbert Correspondence, given some examples, define intersection cohomology sheaves, state the BBDG decomposition.

References:[1], [8], [2]

§6: Introduction to the Affine Grassmannian

Content: Define the affine Grassmannian Gr_G for an algebraic group G. Work out the $\mathbb{C}$-points in some examples. Describe it as an ind-scheme. Give its properties e.g. if G is reductive then it is ind-projective. Describe the connected components of Gr_G. Give the Cartan decomposition. Define Schubert cells from this. If there is time, explain why the Cartan decomposition is a stratification and given properties of the Schubert cells Gr^λ_G i.e. $\overline{\mathrm{Gr}^\lambda_G}$ is an irreducible projective variety of dimension $\langle 2\rho, \lambda\rangle$ (ρ is the Weyl vector aka half sum of positive roots).

References: [9], [1], [2], [7]

§7: Definition and Properties of the Satake Category

Content: All of the sources listed below are useful and you should pick one to follow. The goals of such a talk would be to define the Satake category, but the first part of the talk should wrap up some more details from the previous one e.g. recall Beauville-Laszlo descent and explain what that means for the affine Grassmannian. In describing the Satake category, you will need to explain that $G_{\mathcal{O}}$-equivairances means. You should also describe the simple perverse $G_{\mathcal{O}}$-equivariant sheaves (this will be needed for future speakers). Please make sure to provide some proof e.g. how the IC-sheaves on $G_{\mathcal{O}}$-equivariant.

References: [6], [7], [2], [1], [9]

§9: On construction of a fibre functor $\mathrm{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, k) \to \mathrm{Vec}_k$

Content: Work through the dimension estimates as in §5 of [2] and define the weight functors.

References: [2] but also see [9] and [6].

§10: Why is $\mathrm{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, k)$ a symmetric monoidal category?

Content: Define the convolution product on the Satake category. Explain why it makes the Satake category a symmetric monoidal category.

References: See above.

§11: Towards identifying $\widetilde{G}$ as the Langlands dual group I

Content: This is the difficult part of the proof. Recall properties of the group $\widetilde{G}$ that can be observed from the Satake category. For this talk, focus on why $\widetilde{G}$ is algebraic, split, connected, and reductive.

§12: Towards identifying $\widetilde{G}$ as the Langlands dual group II

Content: Finish the proof by determining that root datum as follows. First, explain why $T_k^\vee$ is the split maximal torus of $\widetilde{G}$. Then explain how to determine $(\widetilde{G}, T_k^\vee)$'s root datum and why it is that of the Langlands dual group.

References: See the ones above. Following [2] and [6] might be best.

§13: General Coefficients

Content: Explain how to extend the above approach to the case where k is a general coefficient (k an arbitrary commutative Noetherian ring of finite global dimension). There are a lot of subtleties here and it might be best to focus on the key statements. It seems Proposition 11.1 of [6] is key and one can refer to §13.1 of [2] for more detials.

References: See above.

References

[1] P.N. Achar. *Perverse Sheaves and Applications to Representation Theory*. Mathematical Surveys and Monographs. American Mathematical Society, 2021. ISBN: 9781470455972. URL: https://books.google.com/books?id=w0ZFEAAAQBAJ.

[2] Pierre Baumann and Simon Riche. *Notes on the geometric Satake equivalence*. 2018. arXiv: 1703.07288 [math.RT]. URL: https://arxiv.org/abs/1703.07288.

[3] P. Deligne and J.S. Milne. *Tannakian Categories*. URL: https://www.jmilne.org/math/xnotes/tc.pdf.

[4] G.A. Jones and J.P. Serre. *Complex Semisimple Lie Algebras*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2000. ISBN: 9783540678274. URL: https://books.google.com/books?id=7AHsSUrooSsC.

[5] J.S. Milne. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2022. ISBN: 9781009018586. URL: https://www.jmilne.org/math/Books/iAG2017.pdf.

[6] I. Mirkovic and K. Vilonen. *Geometric Langlands duality and representations of algebraic groups over commutative rings*. 2018. arXiv: math/0401222 [math.RT]. URL: https://arxiv.org/abs/math/0401222.

[7] Timm Peerenboom. *The Affine Grassmannian with a View Towards Geometric Satake*. URL: https://www.math.uni-bonn.de/ag/stroppel/thesis_timm_peerenboom.pdf.

[8] K. Takeuchi, R. Hotta, and T. Tanisaki. *D-Modules, Perverse Sheaves, and Representation Theory*. Progress in Mathematics. Birkhäuser Boston, 2007. ISBN: 9780817643638. URL: https://books.google.com/books?id=xUHwypdk_aAC.

[9] Xinwen Zhu. *An introduction to affine Grassmannians and the geometric Satake equivalence*. 2016. arXiv: 1603.05593 [math.AG]. URL: https://arxiv.org/abs/1603.05593.