UW–Madison Putnam Club

October 30, 2024 — Geometry

1. Let M be a set of $n \geq 3$ points in the plane such that any three points in M can be covered by a disk of radius 1. Show that the entire set M can be covered by a disk of radius 1. [Canada National Olympiad 2009]

2. Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed into that rectangle (without the interiors of the squares overlapping). You may assume that the sides of the squares will be parallel to the sides of the rectangles. [Putnam 1996 A1]

3. Let L_1 and L_2 be distinct lines in the plane. Prove that L_1 and L_2 intersect if and only if, for every real number $\lambda \neq 0$ and every point P not on L_1 or L_2, there exist points A_1 in L_1 and A_2 in L_2 such that $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$. [Putnam 2017 B1]

4. Consider a convex polyhedron whose faces are triangles and whose edges are oriented. A singularity is a face whose edges form a cycle, a vertex that belongs only to incoming edges, or a vertex that belongs only to outgoing edges. Show that the polyhedron has at least two singularities. [*Putnam and Beyond*]

5. Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose P is a point lying between A and B on this hyperbola such that the area of the triangle APB is maximal. Show that the region between bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB. [Putnam 2015 A1]

6. Is it possible to place infinitely many points in the plane in such that all pairwise distances are integers and the points are not collinear? [Erdős–Anning]

7. Let $\mathcal{T}$ be the set of all triples (a, b, c) of positive integers for which there exists a triangle with side lengths a, b, c. Express
$$\sum_{(a,b,c)\in\mathcal{T}} \frac{2^a}{3^b 5^c}$$
as a rational number in lowest terms. [Putnam 2015 B4]

8. What is the maximum number of rational points that can be on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.) [Putnam 2008 B1]

Hints

1. Helly's theorem is a fun result leading to a neat proof.
4. Euler's formula can play a role.
5. Find a transformation of $\mathbb{R}^2$ that preserves area and the hyperbola.
6. Fix three noncollinear points in the set. Where can remaining points lie?
7. Try to sum first over a, then b, then c (or similar).
8. Given a rational triangle, express the center rationally.