NUMBER THEORY (11/06/19)

Warm-up

1. Let (x, y, z) be a solution to $x^2 + y^2 = z^2$. Show that one of the three numbers is divisible (a) by 3 (b) by 4 (c) by 5.

2. The next to last digit of 3^n is even.

3. What is the last digit of the 2019'th Fibonacci number? (The Fibonacci sequence is defined by $a_1 = a_2 = 1$, and then $a_{k+2} = a_k + a_{k+1}$.)

4. For any $n > 0$, 2^n does not divide $n!$. (Extra question: can you find all n such that 2^{n-1} divides $n!$)

Actual competition problems

5. (VT 2013, 4) A positive integer n is called special if it can be represented in the form

$$n = \frac{x^2 + y^2}{u^2 + v^2},$$

for some positive integers x, y, u, and v. Prove that
(a) 25 is special;
(b) 2013 is not special;
(c) 2014 is not special.

6. (2010-A1) Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.]

7. (2006-A3) Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

8. (2009-B3) Call a subset S of $\{1, 2, ..., n\}$ *mediocre* if it has the following property: Whenever a and b are elements of S whose average is an integer, that average is also an element of S. Let $A(n)$ be the number of mediocre subsets of $\{1, 2, ..., n\}$. [For instance, every subset of $\{1, 2, 3\}$ except $\{1, 3\}$ is mediocre, so $A(3) = 7$.] Find all positive integers n such that

$$A(n + 2) - 2A(n + 1) + A(n) = 1.$$

9. (2008-B4) Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), ..., h(p^2 - 1)$ are distinct modulo p^2. Show that $h(0), h(1), ..., h(p^3 - 1)$ are distinct modulo p^3.

A few important facts from number theory

Standard Conventions. $a|b$ means 'a divides b', $a \equiv b \mod n$ means 'a is congruent to b modulo n, that is, $n|(a-b)$ (or equivalently, a and b have the same remainder when divided by n).

The Chinese Remainder Theorem. If m and n are coprime, then for any a and b there exists a number x such that

$$\begin{cases} x \equiv a \mod m \\ x \equiv b \mod n, \end{cases}$$

moreover, x is unique modulo mn.

Fermat's Little Theorem. If a is not divisible by a prime p, then $a^{p-1} \equiv 1 \mod p$. (Version: for any a and any prime p, $a^p \equiv a \mod p$.)

Euler's Theorem. For any number n, let $\phi(n)$ be the number of integers between 1 and n that are coprime to n. Then for any a that is coprime to n, $a^{\phi}(n) \equiv 1 \mod n$.

Suppose a rational number b/c is a solution of the polynomial equation $a_n x^n + \cdots + a_0 = 0$ whose coefficients are integers. Then $b|a_0$ and $c|a_n$, assuming b/c is reduced.

If $p(x)$ is a polynomial with integer coefficients, then for any integers a and b, $(b-a)|(p(b)-p(a))$.

A number $n \geq 1$ can be written as a sum of two squares if and only if every prime p of the form $4k+3$ appears in the prime factorization of n an even number of times.

Proposed problems for the next meeting (November 13)

1. The last 2019 digits of an integer a are the same as the last 2019 digits of a^2. How many possibilities are there for these 2019 digits?

2. (2007-B1) Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.

3. (2009-B1) Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}$$

4. (2005-A1) Show that every positive integer n is a sum of one or more numbers of the form $2^r 3^s$, where r and s are non-negative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

5. (2013-A2) Let S be the set of all positive integers that are not perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdots a_1 \cdot a_2 \ldots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S onto the integers is one-one.

6. (2008-A3) Start with a finite sequence $a_1, a_2, \ldots, a_n$ of integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$ respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)

7. (1997-B5) Define $d(n)$ for $n \geq 0$ recursively by $d(0) = 1$, $d(n) = 2^{d(n-1)}$. Show that for every $n \geq 2$,

$$d(n) \equiv d(n-1) \mod n.$$

8. (2009-B6) Prove that for every positive integer n, there is a sequence of integers $a_0, a_1, \ldots, a_{2009}$ with $a_0 = 0$ and $a_{2009} = n$ such that each term after a_0 is either an earlier term plus 2^k for some nonnegative integer k, or of the form $b \mod c$ for some earlier terms b and c. [Here $b \mod c$ denotes the remainder when b is divided by c, so $0 \leq (b \mod c) < c$.]