The Game of Nim

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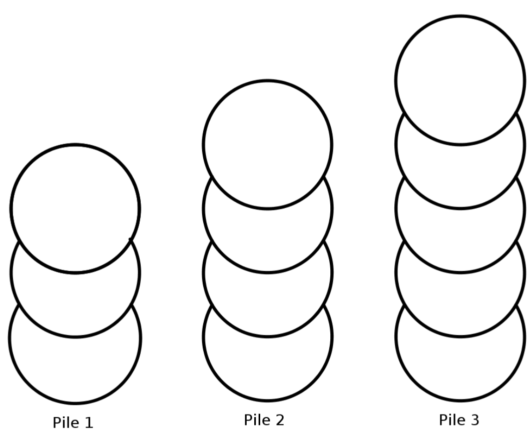
The Madison Math Circle is an outreach organization seeking to show middle and high schoolers the fun and excitement of math! For more information about the Madison Math Circle as well as solutions to these exercises please visit our website at:

`https://www.math.wisc.edu/wiki/index.php/Madison_Math_Circle`.

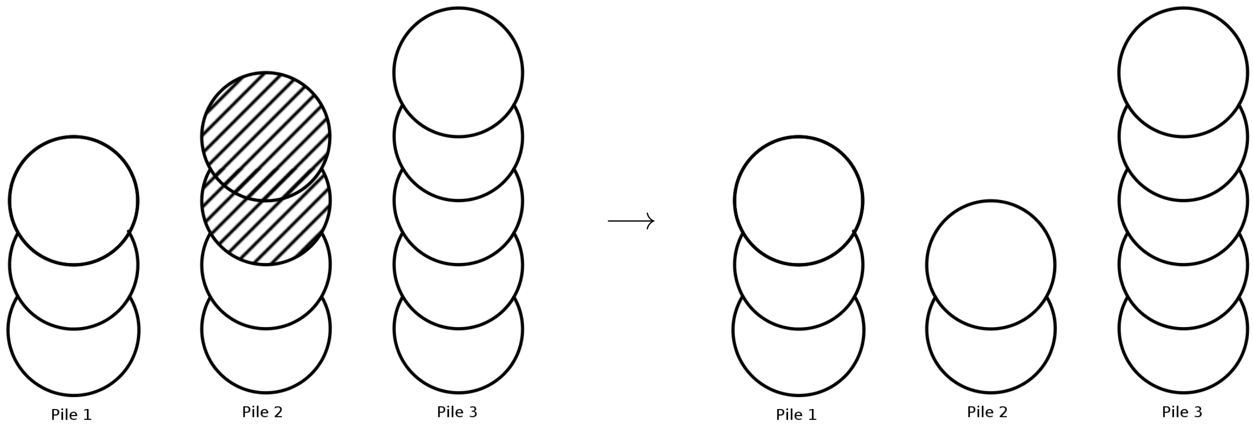
The Game:

Nim is a two-player game played with several piles of stones. You can use as many piles and as many stones in each pile as you want, but in order to better understand the game, we'll start off with just a few small piles of stones. The two players take turns removing stones from the game. On each turn, the player removing stones can only take stones from one pile, but they can remove as many stones from that pile as they want. If they want, they can even remove the entire pile from the game! The winner is the player who removes the final stone.

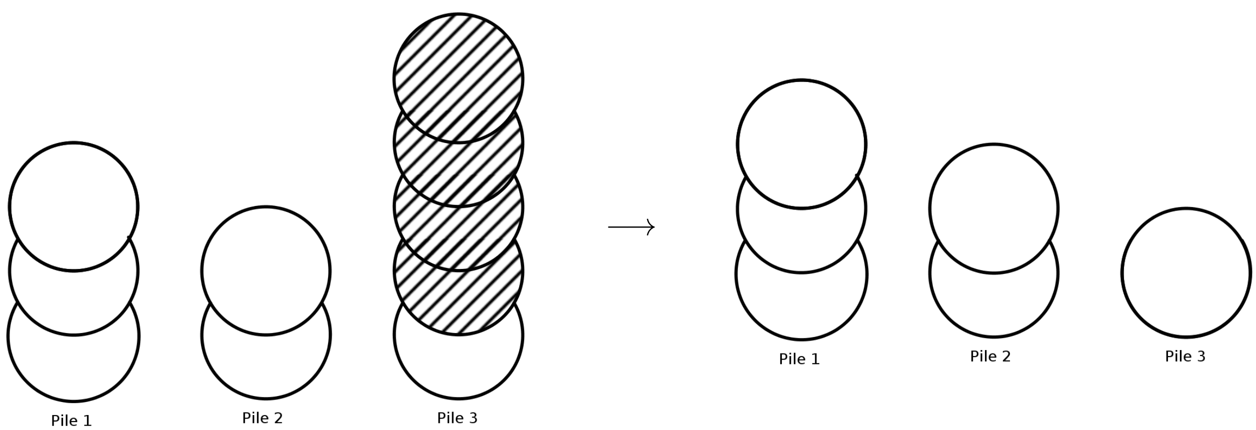
Let's try an example using piles of 3, 4, and 5 stones, as shown below. We'll call the two players Alice and Bob, and for this example, Alice will play first.



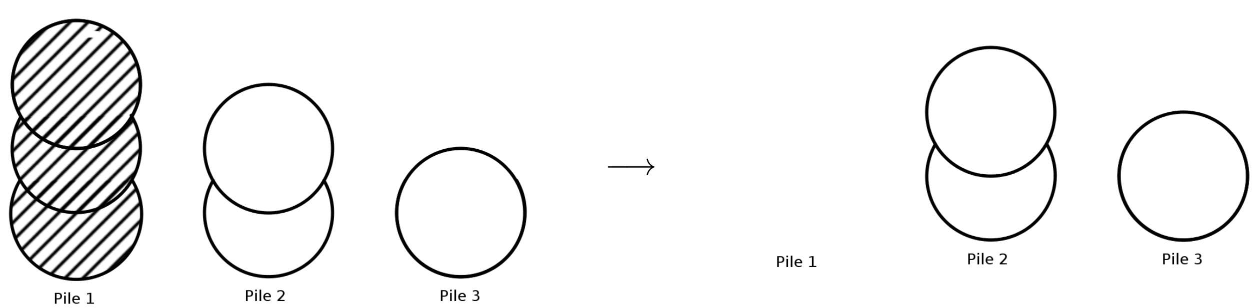
On her first turn, Alice decides to remove 2 stones from pile 2. This leaves Bob with piles of 3, 2, and 5 stones.



On Bob's first turn, he could then decide to take 4 stones from pile 3. This leaves Alice with piles of 3, 2, and 1 stones.

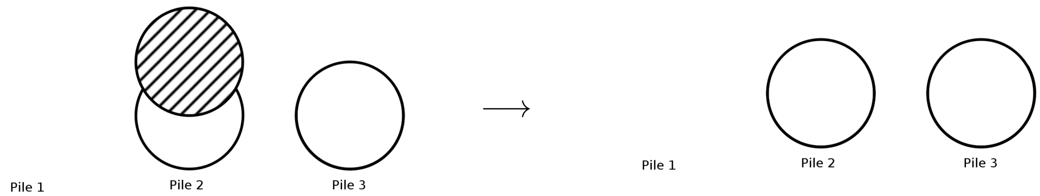


On Alice's next turn, she decides to remove the entire first pile of stones! Now Bob only has two piles left to choose from. On his next turn, he only has three options. He can either remove one stone from pile 2, remove both stones from pile 2, or remove the last stone from pile 3.



Exercise 1. From Bob's current position, he can guarantee victory! Can you figure out what moves he should make to win the game?

Solution. If Bob removes either both stones in pile 2 or the stone from pile 3, he will probably lose, since Alice could win the game by just taking every stone from whatever pile remains. So Bob should take just one stone from pile 2. This would leave two piles with just one stone each. Alice has to take one of these on her turn, and then Bob will win when he takes the last stone on his next turn.



□

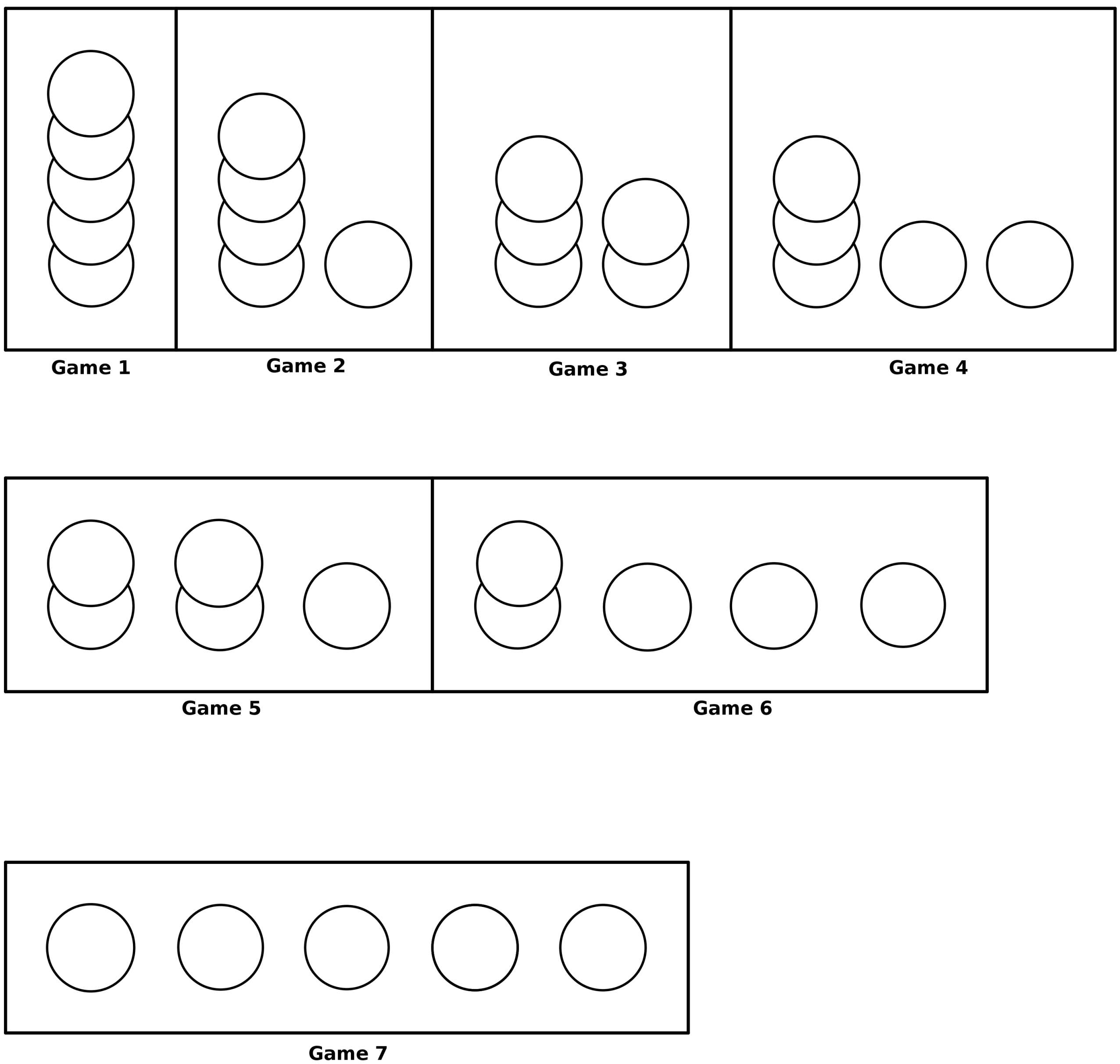
From Playing to Winning:

Nim is an example of an impartial game with perfect information. This means that both players know everything about the current state of the game (in this case, how many stones are left in each pile), and the moves that a player can make depend only on the current position of the game, not on which player is moving (in other words, the only difference between player 1 and player 2 is who goes first). For games of this sort, one of the two players always has a winning strategy! This means that once a game of Nim is set up with some number of piles and some number of stones in each pile, one of the two players can guarantee that they'll win as long as they make the correct sequence of moves. But finding that correct sequence of moves is not always easy! First, the player with the winning strategy depends on how many stones are in each pile. And even if we know who has a winning strategy, that player has to figure out the correct response to any move that their opponent might make!

Exercise 2. Before we start any rigorous analysis, it's good to get more familiar with the game and start developing some ideas about what makes a good move. Try playing several small games of Nim with a friend. What sorts of strategies seem to work better? Can you find any positions near the end of the game that you know how to win from?

Exercise 3. How many different games of Nim can be set up with only 5 stones? For each of these games, can you figure out who has a winning strategy?

Solution. There are 7 different games of Nim using 5 stones. Here are the games:



Player 1 has a winning strategy for all of these games! In game 1, the first player can just take all of the stones immediately. In games 2, 3, 4, and 5, the first player should use his first move to leave his opponent with two piles of the same size, and then mirror the opponents moves for the rest of the game (this will be explained in more detail in exercise 4). In games 6 and 7, the first player should use his first move to leave his opponent with four piles with one stone each; since they each can only take one stone for each of the next four turns, player 1 will win. □

One common (and extremely useful) strategy for figuring out a general winning strategy for a game is to begin analysing a very restricted version of that game first. One feature that makes Nim complicated is that we can have a very large number of different piles of stones. So rather than trying to figure out the whole story at once, let's focus on games of Nim that have only two piles.

Exercise 4. Suppose that the two piles have the same number of stones. Begin by experimenting with two piles of just 2, 3, or 4 stones, and see if you can figure out a winning strategy for one of the two players. Which player has the winning strategy in this case? Can you describe what their strategy would be if each of the piles had 10 stones? What if each of the piles had n stones?

Solution. If the two stones have the same number of stones in them, then the second player has a winning strategy. On each move, if the first player removes some number of stones from one of the two piles, then the second player should respond by removing exactly the same number of moves from the other pile. This will prevent the first player from ever removing the last stone, so the second player will eventually win. Once the first player has removed the last stone from one of the piles, the second player will win by removing all the remaining stones from the other pile. □

Exercise 5. What if the two piles have a different number of stones in them? Who has the winning strategy now?

Solution. If there are a different number of stones in each pile, then the first player has a winning strategy. On the first player's first move, he should remove some stones from the larger pile in order to leave the same number of stones in each pile. With this first move, he takes over the position that the second player had in exercise 4, and the same winning strategy described above will now allow the first player to win. □

Sometimes, when we analyze a special case of a game, we realize that the strategy we developed will work (with minor changes) for other versions of the game as well.

Exercise 6. By using your strategy from exercise 4, can you come up with a winning strategy for a game of Nim that starts with 4 piles of the same size? What about 6 piles of the same size? How would your strategy generalize to a game of Nim with $2n$ piles of the same size?

Solution. If there are an even number of piles of the same size, then the second player has a winning strategy. The idea is to pair up all of the piles, and apply the winning strategy from exercise 4 to each pair of piles independently of the other piles. In other words, every time player 1 removes some number of stones from a pile, player 2 should remove the same number of stones from its partner pile. □

The strategy developed in the previous few exercises doesn't easily generalize to versions of Nim with more complicated starting positions. In particular, it's hard to see how it might apply to our original example where we had 3 piles with 3, 4, and 5 stones.

Exercise 7. The first player has a winning strategy when the game starts with piles of 3, 4, and 5 stones. Now that you know more about what certain winning positions look like, can you figure out the first player's winning strategy? You might begin by trying to find games of Nim with 3 smaller piles of stones where the second player would have a winning strategy, and then figure out how the first player can make sure he leaves the game in those positions after his move.

Solution. The first player should begin by removing 2 stones from the pile with 3 stones. After his opponent's next move, player 1 should remove however many stones is necessary to leave their opponent with either piles of 1, 2, and 3 stones, three piles of 1 stone each, or two piles with the same number of stones. The reader should check that regardless of what move their opponent makes, it is always possible to leave on of these positions. On the remaining moves, the first player should continue trying to leave their opponent with one of the positions in the above list, eventually using the strategy from problem 4 once there are only two piles left. □

The full strategy for Nim was discovered by the mathematician Charles Bouton in 1902. His paper on Nim is considered to be the birth of combinatorial game theory, and it is a particularly interesting application of the binary number system. By continuing to study small examples, like the ones from the exercises above, you may be able to figure out Bouton's strategy yourself!

Variations on a Theme:

There are a wide variety of other combinatorial games that are very similar to Nim, but with minor rule changes. The strategies that we used above (starting with small cases, trying to fully analyse a simpler version of the game, and trying to generalize our strategy from the simpler cases to the full game) are very useful for studying these games as well. In some cases, using those ideas can very quickly lead to the full winning strategy!

Exercise 8. In misére Nim, the rules of play are exactly the same as Nim, but instead of winning by taking the final stone, the winner is the one who forces their opponent to take the final stone by leaving just one stone after their turn. Try following the same outline we used for Nim to figure out some winning strategies for misére Nim. Can you give a concise description of how the winning strategies should change?

Solution. In misére Nim, only the last moves of the winning strategies for Nim change. In particular, we want to leave the opponent with exactly one stone left, so instead of trying to leave two piles of 1 stone each near the end of the game, we should leave either 3 piles of 1 stone each or two piles of 2 stones each. □

Exercise 9. In the game Countdown, we start with just one large pile of stones (often 100). Instead of taking as many stones as we want as we did in Nim, in Countdown, you can only remove up to 10 stones at a time. Can you figure out the winning strategy for Countdown? If you start with 100 stones, does the first or the second player have a winning strategy?

Solution. The first player has a winning strategy when starting with 100 stones. On each move, the first player should remove how ever many stones is necessary to leave their opponent with a multiple of 11 (Check that this is always possible!). Eventually the second player will be left with 11 stones, and they'll have to remove some number of stones that allows player 1 to win on the next turn. □

Exercise 10. Wythoff's game is set up like a game of Nim with two piles of stones, but instead of only being able to remove stones from one pile at a time, the players are also allowed to remove stones from both piles, as long as they remove the same number of stones from each pile. Can you figure out any winning strategies for Wythoff's game?

Solution. See either Wythoff's 1907 paper analyzing this game or the Wikipedia entry for Wythoff's game for a description of a recursive rule that allows one to determine which positions are winning positions for the first player. As in many combinatorial games, a winning strategy can be given by producing the list of all winning and losing positions smaller than the starting state, and then determining on each move which subsequent losing position can be given to the opponent. □

Further Reading:

The game of Nim plays a particularly important role in the study of combinatorial games. By creating a new number system called the "nimbers", games of Nim have become the measuring stick by which positions in all impartial combinatorial games (under normal play) are measured. In fact, the Sprague-Grundy Theorem from the 1930's shows that every impartial game (under normal play) is equivalent to a game of Nim. To learn more about combinatorial games, try reading the series "Winning Ways for Your Mathematical Plays" by Berlekamp, Conway, and Guy or the book "On Numbers and Games" by John Conway. Although these are books about playing games, they get into some very serious mathematics, so don't be intimidated if you don't understand everything the first few times you read it!