Mock Putnam Solutions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that there is some real number a such that $f(f(f(a))) = a$. Show that there is some real number b such that $f(b) = b$.

Solution: Consider $g(x) = f(x) - x$. If $g(x)$ has a root b, then $f(b) = b$ and we are done. Suppose for contradiction that it does not. If $g(x)$ takes both positive and negative values, then by the intermediate value theorem it has a root. Otherwise, $g(x)$ is everywhere positive or everywhere negative. In the former case, this gives $f(x) > x$ everywhere, so $f(f(f(a))) > f(f(a)) > f(a) > a$, contradicting the fact that $f(f(f(a))) = a$. In the latter case, we similarly have $f(x) < x$ and thus $f(f(f(a))) < a$, again a contradiction. In every case we have a contradiction, so in fact $g(x)$ has a root and thus there is a b with $f(b) = b$.

2. Let $g(n)$ be the number of ways to write n as the ordered sum of positive integers, at least one of which is even and at least one of which is odd. Find, with proof, $g(11)$ and $g(12)$.

Solution: It is easiest to calculate $g(n)$ by finding $f(n)$, then number of all such sums, $h(n)$, the number of such sums composed of only odd integers, and $k(n)$, the number of such sums composed of only even integers. Then $g(n) = f(n) - h(n) - k(n)$.

We begin by computing $f(n)$. Write $n = 1 + 1 + 1 + \cdots + 1$. Then for each '+' in this sum, we can choose to either leave it or to erase it and add the numbers on either side. Each way of doing this gives a different ordered sum, and conversely, every ordered sum can be obtained in this way. There are $n - 1$ choices, so $f(n) = 2^{n-1}$.

Next, observe that $k(n) = 0$ if n is odd. If n is even, then by dividing each term in the sum by 2, we see that $k(n) = f(\frac{n}{2}) = 2^{\frac{n}{2}-1}$.

Finally, we consider $h(n)$. Let $n > 2$ and consider an ordered sum of odd integers adding to n. There are two cases. For the sums ending in a 1, removing the 1 gives a bijection with all ordered sums of odd integers adding to $n - 1$. For the sums ending in an integer greater than 1, subtracting 2 from the last summand gives a similar bijection with ordered sums of odd integers adding to $n - 2$. Hence $h(n) = h(n-1) + h(n-2)$. Since $h(1) = h(2) = 1$, these $h(n)$ are the Fibonacci numbers.

Thus, $g(11) = f(11) - h(11) - k(11) = 1024 - 89 - 0 = 935$, and $g(12) = f(12) - h(12) - k(12) = 2048 - 144 - 32 = 1872$.

3. Let $P(x) = x^{100} + 20x^{99} + 198x^{98} + a_{97}x^{97} + \cdots + a_1 x + 1$ be a polynomial, where the a_i $(1 \leq i \leq 97)$ are real numbers. Prove that $P(x) = 0$ has at least one complex root (i.e., a root of the form $a + bi$ with a, b real numbers and $b \neq 0$). (VTRMC 2011)

Solution: Let the roots be r_i, for $1 \le i \le 100$. Observe that $-20 = \sum_{i=1}^{100} r_i$, as well as $198 = \sum_{1\le i<j\le 100} r_i r_j$. Hence $\sum_{i=1}^{100} r_i^2 = (-20)^2 - 2(198) = 4$. Furthermore, we also have that $\prod_{i=1}^{100} r_i^2 = \left(\prod_{i=1}^{100} r_i\right)^2 = 1$.

Suppose for contradiction that all the roots r_i were real. Then their squares r_i^2 would all be nonnegative. Thus, by the AM-GM (arithmetic mean - geometric mean) inequality, we have

$$1 = \sqrt[100]{\prod_{i=1}^{100} r_i^2} \le \frac{1}{100}\sum_{i=1}^{100} r_i^2 = \frac{1}{25}.$$

This is a contradiction, so it follows that at least one r_i is complex.

4. Find, with proof, the number of ordered pairs of integers (m, n) such that $\frac{1}{m}+\frac{1}{n} = \frac{1}{91}$.

Solution: Multiply through by $91mn$ and rearrange to get $mn - 91m - 91n = 0$. Adding 91^2 to both sides and factoring, we obtain $(m - 91)(n - 91) = 91^2$. So for every solution (m, n) we have a different ordered pair of integers which multiply to 91^2. Now $91^2 = 7^2 \cdot 13^2$ has exactly 9 positive divisors, so exactly 18 total divisors, and each one corresponds to exactly one such ordered pair (namely, the one where the chosen divisor is the first term in the product). Each of these yields an ordered pair (m, n) (given by adding 91 to the divisor and its complement) satisfying the original equation, provided that neither m nor n thus obtained is zero. This problem occurs when the chosen divisor is -91, and only then, so there are exactly 17 solutions (m, n) to the original equation. (It is straightforward to compute them and check that they work if one so desires.)

5. Find, with proof, all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with continuous derivative such that $xf(x) = f(x^2)$ holds for all $x \in \mathbb{R}$.

Solution: Clearly $f(x) = kx$ for a constant k satisfies the given equation. We will show that these are all the solutions.

First Solution: Iterate the given equation starting with $f(x^{2^n})$ to obtain that $(x^{2^n-1})f(x) = f(x^{2^n})$. Taking derivatives on both sides yields

$$(2^n - 1)x^{2^n-2}f(x) + x^{2^n-1}f'(x) = 2^n x^{2^n-1}f'(x^{2^n}).$$

Rearranging (provided $x \ne 0$), we get

$$f(x) - xf'(x^{2^n}) = \frac{1}{2^n}(f(x) - xf'(x)).$$

Let $c > 0$, and take $x_n = \sqrt[2^n]{c}$, so that we get

$$f(x_n) - x_n f'(c) = \frac{1}{2^n}(f(x_n) - x_n f'(x_n)).$$

Now $x_n \to 1$ as $n \to \infty$, so taking the limit of the above equation and using the fact that f and f' are continuous, we obtain that $f(1) - f'(c) = 0$. So $f'(x)$ is constant (and equal to $f(1)$) on $x > 0$. Thus $f(x) = xf(1)$ for positive x. Also, from the original equation, $f(0) = 0$ and $-xf(-x) = f((-x)^2) = f(x^2) = xf(x)$, so $f(-x) = -f(x) = -xf(1)$ for x positive. Hence in fact $f(x) = xf(1)$ for all values of x, as required.

Second Solution: Define $g(x) = \dfrac{f(x)}{x}$ for $x \neq 0$ and $g(0) = f(1)$. Thus, $xg(x) = f(x)$ for all x. Since $f(x)$ is continuous, $g(x)$ is continuous except possibly at 0. Now, $x^2 g(x) = xf(x) = f(x^2) = x^2 g(x^2)$, whence $g(x) = g(x^2)$ for $x \neq 0$. Now for $x > 0$, define $x_n = \sqrt[2^n]{x}$ and observe that the preceding gives $g(x) = g(x_n)$ for all n. Since g is continuous at 1 and $\lim_{n\to\infty} x_n = 1$, this gives that $g(x) = g(1)$ for all $x > 0$. Additionally, $g(0) = f(1) = g(1)$, and $g(x) = g(x^2) = g(1)$ for negative x as well. So $g(x) = g(1) = f(1)$ is a constant, and hence $f(x) = xg(x) = xg(1)$ is of the form claimed. (NB: The differentiablity of $f(x)$ was not required, only its continuity.)