Math 222 - Midterm 2 Solutions

Problem 1: (12 points)

(3 points each) No partial credit was awarded for this question, since you did not need to show your work. If you put contradictory answers, you were not awarded any points. In particular, if you said the limit exists and equals $\pm\infty$, that was marked as wrong.

a. The limit does not exist, and does not go to ∞ or $-\infty$.

b. The limit does not exist, and does not go to ∞ or $-\infty$.

c. The limit exists and equals 0.

d. The limit does not exist and goes to ∞.

Problem 2: (6 points)

(3 points each)

a. The statement here is exactly the function rule taught in class. The correct answer is True.

b. This question asked whether the opposite of the function rule is true. The correct answer is False.

   To see this in more detail, consider the function $f(x) = \sin(2\pi x)$. Since this is a sin function, $\lim_{x\to\infty} f(x)$ does not exist. However, if we plug in any positive integer n, we get $f(n) = \sin(2\pi n) = 0$. So, if $a_n = f(n)$, then $a_n = 0$. We then get $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = 0$.

Problem 3: (15 points)

(5 points) Let $f(x) = \sin(\pi + \frac{x}{3})$, so $f(0) = \sin \pi = 0$. Then

$$\begin{aligned}
f'(x) &= \frac{1}{3}\cos(\pi + x/3) \implies f'(0) = \frac{1}{3}\cos \pi = -\frac{1}{3} \\
f''(x) &= -\frac{1}{9}\sin(\pi + x/3) \implies f''(0) = -\frac{1}{9}\sin \pi = 0 \\
f'''(x) &= -\frac{1}{27}\cos(\pi + x/3) \implies f'''(0) = -\frac{1}{27}\cos \pi = \frac{1}{27} \\
f^{(4)}(x) &= \frac{1}{81}\sin(\pi + x/3) \implies f^{(4)}(0) = 0.
\end{aligned}$$

(3 points) Now we see the general pattern:

$$f^{(n)}(0) = \begin{cases} 0 & \text{n even} \\ -3^{-n} & \text{n odd, } n = 1, 5, 9, ... \\ 3^{-n} & \text{n odd, } n = 3, 7, 11, ... \end{cases} .$$

(2 points) The formula for the 16th Taylor polynomial is

$$T_{16}f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^{(16)}(0)x^{16}}{16!}.$$

(4 points) Plugging in the values we found above,

$$T_{16}f(x) = -\frac{x}{3} + \frac{x^3}{3^3 \cdot 3!} - \frac{x^5}{3^5 \cdot 5!} + \frac{x^7}{3^7 \cdot 7!} - \frac{x^9}{3^9 \cdot 9!} + \frac{x^{11}}{3^{11} \cdot 11!} - \frac{x^{13}}{3^{13} \cdot 13!} + \frac{x^{15}}{3^{15} \cdot 15!}$$

(1 point was awarded for stopping at the correct spot)

If you noticed that $g(x) = \sin(x/3)$, gave the correct 16th Taylor polynomial for $\sin(\pi + \frac{x}{3})$, and then gave the correct answer to the problem (without justification as to why it was the requested Taylor polynomial), 5 points partial credit were awarded.

If you noticed that $\sin(\pi + x/3) = \sin(x/3)$, and then determined the correct 16th Taylor polynomial of $-\sin(x/3)$ with justification as to why it was the Taylor polynomial, (either analogous to the above or otherwise), 15 points were awarded.

Problem 4: (20 points)

(a) The correct answer was neither, as it is not separable, nor is it first-order linear. **(2 points)**

(b) The correct picture was the third. One way you could see this is by figuring out what the picture should look like on the x and y axes. **(4 points)**

(c) 2 points were awarded for plotting the right initial points, (-4,3) and (0,3). 2 points were awarded for the correct curves. Points were still awarded if they chose the wrong direction field in part (b). The curves had to extend in both directions in order to get full credit. **(4 points)**

(d) Fully correct answers had 3 main ideas in them. 2 points were given for writing that the short lines in the direction field represent tangent lines to the graph of the solution passing through that point. 4 points were then given for mentioning that tangent lines are very close to the actual curve over a short distance. 4 points were given for writing that after the first tangent line you need to find a new approximate tangent line. This occurs because when the step size is large, the old tangent line is no longer a good approximation of the curve. No points were given if the student only said that they were following the slope or flow starting from the initial points. **(10 points)**

Problem 5: (20 points)

(a) **(6 pts)**

4 pts (1 pt partial credit if only $(1.48)^{10}$ **and** 100 **were wrong)**

$$B(t) = 740(1.48)^{t/10} - 100$$

The bacteria start at a population of 740 and increase by 48% each 10 days, and then *at the end of the t days* (2 **pts** for explaining this about an equation like that above), after all the growing is done, 100 bacteria are removed.

Another way to understand the exponential growth term is to say: for what r is $1000r^{10} = 1480$? In other words, what is the daily growth factor r that explains an increase from 1000 to 1480 in 10 days? We solve and find $r = (1.48)^{1/10}$. So after starting with a population of 740, after t days but before removing any bacteria we have $740((1.48)^{1/10})^t$ bacteria.

(b)**(14 pts)**

9 pts (2 pts partial credit if only $\ln((1.48)^{1/10})$ **and** -24 **were wrong, OR 5 pts partial credit for all correct except with a daily growth rate of** 4.8%**)**

$$\frac{dD}{dt} = \ln((1.48)^{1/10})D - 24$$

5 pts for the explanation of both terms above, which required that the differential equation given had these features to explain The change in the population comes from two sources, the natural growth, and the removal for diagnostic tests. The removal happens at a rate of 24 bacteria per day, close to continuously (1 each hour) throughout the day. The -24 term above represents this constant negative contribution to the rate of change $\frac{dD}{dt}$ of the number of bacteria.

From part (a) above, we see without removal of any bacteria, the natural birth and death of the bacteria lead to a population

$$P(t) = 740(1.48)^{t/10}$$

at time t. We can calculate

$$\frac{dP}{dt} = 740\ln(1.48)\frac{1}{10}(1.48)^{t/10}.$$

So we see that

$$\frac{dP}{dt} = \ln(1.48)\frac{1}{10}P,$$

and the natural growth of the bacteria is with an instantaneous growth rate of $\ln(1.48)\frac{1}{10} = \ln((1.48)^{1/10})$. This explains the $\ln((1.48)^{1/10})D$ term in the above differential equation. The instantaneous growth of the population is proportional to the current population D, and with proportion $\ln((1.48)^{1/10})$ as we saw above.

Problem 6: (15 points)

This questions asks to find three specific (particular) solutions to the differential equation

$$\frac{dt}{dx} = xy + xe^{x^2}.$$

This is a linear differential equation.

Written in the form $\frac{dt}{dx} - xy = xe^{x^2}$, we have $a(x) = -x$ so $A(x) = -\frac{x^2}{2}$, and we have $k(x) = xe^{x^2}$.

Correctly identifying each part of the linear equation is worth 3 points.

We can now apply our formula for the general solution of a linear differential equation:

$$y = e^{-A(x)} \left(\int e^{A(x)} k(x) dx + C \right).$$

In our case:

$$y = e^{\frac{x^2}{2}} \left(\int e^{-\frac{x^2}{2}} x e^{x^2} dx + C \right).$$

Correctly applying the formula is worth 2 points.

Time to do the integral.

First notice that $e^{-\frac{x^2}{2}} x e^{x^2} = x e^{-\frac{x^2}{2} + x^2} = x e^{\frac{x^2}{2}}$, so we have that

$$y = e^{\frac{x^2}{2}} \left(\int e^{-\frac{x^2}{2}} x e^{x^2} dx + C \right) = e^{\frac{x^2}{2}} \left(\int e^{\frac{x^2}{2}} x dx + C \right).$$

We will make the substitution $u = \frac{x^2}{2}$, so $du = 2\frac{x}{2}dx = xdx$, and

$$\int e^{\frac{x^2}{2}} x dx + C = \int e^u du + C = e^u + c = e^{\frac{x^2}{2}} + C.$$

In conclusion or general solution is

$$y = e^{\frac{x^2}{2}} \left(\int e^{\frac{x^2}{2}} x dx + C \right) = e^{\frac{x^2}{2}} \left(e^{\frac{x^2}{2}} + C \right).$$

The solution can be express as $y = e^{\frac{x^2}{2}} \left(e^{\frac{x^2}{2}} + C \right)$ or as $y = e^{x^2} + Ce^{\frac{x^2}{2}}$.

4 points were given for the integration, 1 of them for writing C in its right place.

To find the three particular solutions we just need to give three values to C. For example we can take $C = -1, C = 0$ and $C = 1$, in which case we obtain:

$$\begin{aligned} y_{-1}(x) &= e^{x^2} - e^{\frac{x^2}{2}}, \qquad & y_{-1}(0) = e^{0^2} - e^{\frac{0^2}{2}} = 1 - 1 = 0. \\ y_0(x) &= e^{x^2}, \qquad & y_0(0) = e^{0^2} = 1. \\ y_1(x) &= e^{x^2} + e^{\frac{x^2}{2}}, \qquad & y_1(0) = e^{0^2} + e^{\frac{0^2}{2}} = 1 + 1 = 2. \end{aligned}$$

This part was worth 6 points. 2 pts for each particular solution and its value at 0.

Those who instead wrote the relation $y(0) = 1 + C$, without picking values for C, were awarded 3 points. The 6 points were awarded even if the general solution was incorrect.

Problem 7: (20 points)

In order to approximate $\ln(2)$, we use a Taylor polynomial. You could use $T_n^1 \ln(x)$ or $T_n^0 \ln(1+x)$. Since we have an explicit formula for the second, I will use that (though the first is valid). I will use $T_4 \ln(1+x)$. We get:

$$T_4 \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

Finding a correct Taylor polynomial that can be used to approximate $\ln(2)$ was worth 3 points. Full points were not awarded for the first order Taylor polynomial, as $T_1 \ln(1+x) = x$, which is not very good for approximating $\ln(2)$.

Using the Taylor polynomial for $\ln(1+x)$, we can plug in $x = 1$ to get an approximation for $\ln(2)$. If you used $T_n^1 \ln(x)$, you would have to plug in $x = 2$. In our case, we get:

$$\ln(2) \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$$

Finding an approximation was worth 2 points.

Next, we want to bound the error. In order to do this, you had to take $n+1$ derivatives of $f(x) = \ln(1+x)$ if you used $T_n \ln(1+x)$, and $n+1$ derivatives of $\ln(x)$ if you used $T_n^1 \ln(x)$. We get:

$$\begin{aligned}
f(x) &= \ln(1+x) \\
f'(x) &= \frac{1}{1+x} \\
f''(x) &= \frac{-1}{(1+x)^2} \\
f^{(3)}(x) &= \frac{2}{(1+x)^3} \\
f^{(4)}(x) &= \frac{-6}{(1+x)^4} \\
f^{(5)}(x) &= \frac{24}{(1+x)^5}
\end{aligned}$$

Doing this correctly was worth 4 points.

We now wish to find an M such that $|f^{(n+1)}(t)| \leq M$ for all $t \in [0,1]$. In the case I am doing, $n = 4$, so we get:

$$\begin{aligned}
|f^{(5)}(t)| &= \left|\frac{24}{(1+t)^5}\right| \\
&\leq \left|\frac{24}{(1+0)^5}\right| = 24
\end{aligned}$$

We let $M = 24$. In general, if you took $T_n \ln(1+x)$, you should have gotten $M = n!$. **Finding a correct M was worth 3 points.**

By the remainder theorem, we then get:

$$|\ln(2) - \frac{7}{12}| \leq \frac{(24)(1)^5}{5!}$$

Correctly appplying the remainder theorem was worth 3 points.

Finally, in the case we have, this bound is $\frac{24}{5!} = \frac{4!}{5!} = \frac{1}{5}$. If you had used $T_n \ln(1+x)$, you should get that the error is at most $\frac{1}{n+1}$. You needed to make n at least 4 in order to ensure the error is at most 0.2.

If you used a large enough n and correctly deduced that the error was at most 0.2, you were awarded the remaining 5 points. If there was a small error in your arithmetic that made your answer wrong, 1 point was taken off.