MATH 240

Midterm #1 · Section 2

OCTOBER 10, 2013

NAME:

GRADE

Instructions:

1. This Midterm consists of six questions. The total points for each of them collected in the table below.
2. Each question must be answered clearly on a **separate sheet of paper**, **ink** and **detail any reasoning used to justify**.
3. **No** notes, books, pagers, cell phones or electronic devices are **allowed**.
4. The duration of this test is **1 hours** and **15 minutes**.

QUESTION	*POINTS*	*SCORE*
1	*15*	
2	*20*	
3	*15*	
4	*15*	
5	*15*	
6	*20*	
	TOTAL	

1. Show that $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ are logically equivalent. [*15 points*]

Solution: Let us construct the truth table for the given compound propositions.
For proposition $(p \to q) \wedge (p \to r)$ we have:

Table 1: Truth Table for $(p \to q) \wedge (p \to r)$

p	q	r	$p \to q$	$p \to r$	$(p \to q) \wedge (p \to r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

while for $p \to (q \wedge r)$,

Table 2: Truth Table for $(p \to q) \wedge (p \to r)$

p	q	r	$q \wedge r$	$p \to (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Since they have the same final column in their respective truth tables we conclude that both propositions are logically equivalents.

2. Construct the truth table for the compound propositions

   (a) $\big[(p \to q) \wedge (\neg p \to q)\big] \to q.$ [*10 points*]

   (b) $\big[(p \to q) \wedge (\neg p \to r)\big] \to (\neg q \to r).$ [*10 points*]

Solution:

(a) The truth table in this case is

Table 3: Truth Table for $\big[(p \to q) \wedge (\neg p \to q)\big] \to q$

p	q	$\neg p$	$p \to q$	$\neg p \to q$	$(p \to q) \wedge (\neg p \to q)$	$\big[(p \to q) \wedge (\neg p \to q)\big] \to q$
T	T	F	T	T	T	T
T	T	F	T	T	T	T
T	F	F	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	T	T	T	T	T
F	F	T	T	F	F	T
F	F	T	T	F	F	T

(b) For the second case,

Table 4: Truth Table for $\big[(p \to q) \wedge (\neg p \to r)\big] \to (\neg q \to r)$

p	q	r	$\neg p$	$\neg q$	$p \to q$	$\neg p \to r$	$(p \to q) \wedge (\neg p \to r)$	$\neg q \to r$	$\big[(p \to q) \wedge (\neg p \to r)\big] \to (\neg q \to r)$
T	T	T	F	F	T	T	T	T	T
T	T	F	F	F	T	T	T	T	T
T	F	T	F	T	F	T	F	T	T
T	F	F	F	T	F	T	F	F	T
F	T	T	T	F	T	T	T	T	T
F	T	F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T	T	T
F	F	F	T	T	T	F	F	F	T

3. (a) For a given integer $d > 1$ let $A_d :\ = \{n \in \mathbb{N} \big| \ d \text{ divides } n\}$. What is the least element in $A_2 \cap A_4 \cap A_5$?

   (i) 2 (ii) 10 (iii) 20 (iv) 40. [*8 points*]

   (b) Consider the sequence of sets $S_1 :\ = \{a\}$, $S_2 :\ = \{b\}$ and recursively defined by $S_i = \{a, b, c\} - (S_{i-1} \cup S_{i-2})$. What is the 7^{th} term of this sequence

   (i) $\{a\}$ (ii) $\{b\}$ (iii) $\{c\}$? [*7 points*]

Solution:

(a) Clearly

$$A_2 = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \dots\}$$
$$A_4 = \{4, 8, 12, 16, 20, \dots\}$$

and

$$A_5 = \{5, 10, 15, 20, \dots\}.$$

Now notice that $20 \in A_2 \cap A_4 \cap A_5$ and that $5, 10, 15 \in A_5$ are the only elements in A_5 smaller that 20 and none of them belong to $A_2 \cap A_4$. Hence, the answer is (iii).

(b) Recursively we have

$$\begin{aligned} S_1 &= \{a\} \\ S_2 &= \{b\} \\ S_3 &= \{a, b, c\} - \{a, b\} = \{c\} \\ S_4 &= \{a, b, c\} - \{b, c\} = \{a\} \\ S_5 &= \{a, b, c\} - \{a, c\} = \{b\} \\ S_6 &= \{a, b, c\} - \{a, b\} = \{c\} \end{aligned}$$

and

$$S_7 = \{a, b, c\} - \{b, c\} = \{a\}.$$

As we can see, in general

$$S_n = \begin{cases} \{c\} & \text{if} \quad n \equiv 0 \mod 3 \\ \{a\} & \text{if} \quad n \equiv 1 \mod 3 \\ \{b\} & \text{if} \quad n \equiv 2 \mod 3. \end{cases}$$

4. The market value of a car low 10% annually. At the time of purchase its value was \$ 40,000.

   (a) Set up the recurrence relation for the current market value after n years. [*8 points*]

   (b) Find its market price three years later after the first purchase. [*7 points*]

Solution:

(a) If υ_n denotes the market value of the car after n years of purchase, then

$$\upsilon_{n+1} = \upsilon_n - 0.1 \cdot \upsilon_n = 0.9 \cdot \upsilon_n,\ n = 0, 1, \ldots$$

according to the assumption where the starting value of the car is $\upsilon_0 = \$\,40,000$.

(b) After three years, the market value of the car will be

$$\upsilon_3 = 0.9\upsilon_2 = 0.9^2\upsilon_1 = 0.9^3\upsilon_0 = 0.9^3 \cdot 40,000 = 0.729 \cdot 40,000 = \$\,29,160.$$

5. Find a 2×2 matrix A such that

$$A \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

[*15 points*]

Hint: Solve a system of linear equations.

Solution: Put

$$A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}.$$

Then

$$A \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 2x + y \\ z & 2z + t \end{pmatrix}.$$

Hence, the given equation is equivalent to the system of linear equations

$$\left.\begin{array}{rcl} x & = & 1 \\ 2x + y & = & 2 \\ z & = & 1 \\ 2z + t & = & 1 \end{array}\right\}$$

whose solution is $x = 1$, $y = 0$, $z = 1$ and $t = -1$ giving

$$A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

6. Prove or give a counterexample

   (a) If a, b, c, m are integers with $m > 1$ and $ac \equiv bc \mod m$, then $a \equiv b \mod m$. [*10 points*]

   (b) If n is an odd integer then $n^2 \equiv 1 \mod 8$. [*10 points*]

Solution:

(a) This is false in general. For example, if $m = 6$, $a = 2$, $b = 0$ and $c = 3$ we have

$$2 \cdot 3 = 6 \equiv 0 = 0 \cdot 3 = 0 \mod 6$$

but $2 \not\equiv 0 \mod 6$.

(b) Any odd integer n can be written as $n = 2\kappa + 1$ for some $\kappa \in \mathbb{Z}$ and so $n^2 = 4\kappa^2 + 4\kappa + 1 = 4\kappa(\kappa + 1) + 1$. Now note that since κ and $\kappa + 1$ are consecutive integers, at least one of them must be even. Therefore $\kappa(\kappa + 1) = 2\ell$ is even and hence $n^2 = 8\ell + 1 \equiv 1 \mod 8$.

Remark. Here is another way to solve this question: since the remainder modulo 8 of any odd integer is again odd, it is enough to check that $n^2 \equiv 1 \mod 8$ for $n = 1, 3, 5, 7$ (which are all possible odd reminders modulo 8). But trivially

$$1^2 = 1 \mod 8$$
$$3^2 = 9 = 1 \mod 8$$
$$5^2 = 25 = 1 \mod 8$$

and finally

$$7^2 = 49 = 1 \mod 8.$$