Math 222 - Midterm 1 Solutions

Problem 1

The two only substitutions that actually give you back an easy integral were a trig substitution (using $x = \tan\theta$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ or $x = \cot\theta$ with $\theta \in (0, \pi)$) and a rational substitution (using $x = \frac{1}{2}(t - \frac{1}{t})$ with $t > 0$).

Any not useful substitution wasn't worth any partial credit. Also, if you have already shown a trig substitution the second one wasn't worth partial credit (since it was not substantially different).

In the explanation, points were given depending if there were two substantially different substitutions or not and if the explanation described an actual way to finish the integral.

Here the sample of a correct answer:

(a) (6 points) Let $x = \tan(\theta)$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (2 pt for recognizing the corect variable, 1 pt for the bounderies of θ), also $dx = \sec^2(\theta)d\theta$ (1 pt for this remark).

Then our integral changes like this:

$$\int (x^2+1)^{\frac{3}{2}} dx = \int (tan^2(\theta)+1)^{\frac{3}{2}} sec^2(\theta) d\theta =$$

$$= \int (sec^2(\theta))^{\frac{3}{2}} sec^2(\theta) d\theta = \int sec^3(\theta)\sec^2(\theta) d\theta = \int \sec^5(\theta) d\theta.$$

(Here putting everything in terms of θ worth 1 pt, use the fact that $tan^2(\theta) + 1 = sec^2(\theta)$ worth 1 pt and the correct change of dx worth another, use $x = \cot(\theta)$ with $\theta \in (0, \pi)$ has the same rubric).

(b) (6 points) Let $x = \frac{1}{2}(t - \frac{1}{t})$ with $t > 0$ (2 pt for stating the right rational substitution [could lose 1 pt if the use the one with +] and 1 pt for the bound of t [we agreed to give the point even if the bound was for $t \geq 1$]), also $dx = \frac{1}{2}(1 + \frac{1}{t^2})dt$ (this remark worth 1 pt).

Then our integral changes to:

$$\int (x^2+1)^{\frac{3}{2}} dx = \int ((\frac{1}{2}(t - \frac{1}{t}))^2 + 1))^{\frac{3}{2}} \frac{1}{2}(1 + \frac{1}{t^2}) dt =$$

$$= \int ((\frac{1}{2}(t+\frac{1}{t}))^2)^{\frac{3}{2}} \frac{1}{2}(1+\frac{1}{t^2})dt = \int (\frac{1}{2}(t+\frac{1}{t}))^3 \frac{1}{2}(1+\frac{1}{t^2})dt.$$

(Here changing everything in terms of t worth 2 pt and make the right change of dx worth 1 pt).

(c) (3 points) The first point of (c) was only achieved if there were two correct substitutions to choose between (a trigonometric and a rational). After that either of the choices was acceptable but it was easy to earn the points with the rational substitution:

Rational Substitution: I would expand (foil) the power of the polynomial to get terms that only have powers of t (positive or negative) I'll integrate that and after that come back to x (2 pt for a description like this, only 1 if the description was vague). [The remaining integral here is very easy to do!]

Trigonometric Substitution: I'll use the reduction formula

$$\int \sec^n(x)dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$$

over the integral two times, at the end I'll use the fact that

$$\int \sec(x)dx = \ln|\sec(x)+\tan(x)| + C.$$

Finally I would go back to x using $\theta = \arctan(x)$ (2 pt for a description like this, only 1 pt if the explanation only talks about manipulating trig identities, using integration by parts or using a reduction formula).

Problem 2: (10 points)

2 points were awarded for stating a valid trigonometric identity. You needed to write it out explicitly.
Ex: $\cos^2(x) = 1 - \sin^2(x)$.

Writing an integral where the identity is applicable was worth 3 points.
Ex: $\int \sin^2(x)\cos^3(x)dx$.

Finally, actually using the trigonometric identity to solve the integral was worth 5 points. Full points were awarded for using the identity you stated, showing work, and writing the integral correctly.
Ex:

$$\begin{aligned}\int \sin^2(x)\cos^3(x)dx &= \int \sin^2(x)\cos^2(x)\cos(x)dx \\ &= \int \sin^2(x)(1-\sin^2(x))\cos(x)dx \\ &= \int (\sin^2(x) - \sin^4(x))\cos(x))dx\end{aligned}$$

Using the substitution $u = \sin(x)$, we know $du = \cos(x)dx$, so the integral becomes:

$$\begin{aligned}\int (u^2 - u^4)du &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C\end{aligned}$$

Problem 3:

(a) (6 points) Substitution: $u = \sin x$ for $x \in [-\pi/2, \pi/2]$ [1 point for good choice of substitution, 1 point for choice of interval for x]

$du = \cos x dx$

$$\int_{-1}^{1} \sqrt{1-u^2}du = \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2 x}\cos x dx = \int_{-\pi/2}^{\pi/2} \cos^2 x dx$$

[2 points for correctly executed (valid, useful) substitution, 2 points for correct bounds] (This is easier to do, e.g. by the half-angle trig identity, but you did not need to say this. There is a similar substitution $u = \cos x$ that works as well.)

(b) (4 points) There are of course many possible answers. Some valid substitutions include $u = x + 1$ or $u = \ln x$, or $u = \tan x$ for $x = [-\pi/4, \pi/4]$. Here is a sample correct answer: Substitution: $u = \tan x$ for $x = [-\pi/4, \pi/4]$

$du = \sec^2 x dx$

$$\int_{-1}^{1} \sqrt{1-u^2}du = \int_{-\pi/4}^{\pi/4} \sqrt{1-\tan^2 x}\sec^2 x dx.$$

(1 point for a choice of valid substitution, 2 points for correctly executed (valid) substitution,1 point for correct bounds)

(Remember that a valid choice should be a $1-1$ function of u for u from -1 to 1. So it is not valid to take, for example, $u = \sec x$ since $\sec x$ does not take any values between -1 and 1 (except for $-1, 1$). One cannot take $x = u^2$ because that is not a $1-1$ function for u from -1 to 1, e.g. when $u = 1$ we have $x = 1$ and when $u = -1$ we also have $x = 1$. See your homework problem from Week 4– Chapter I, Section 15, #44 –where you explained what can go wrong when you make this kind of invalid substitution.)

Problem 4:

(a) (5 points) You could use integration by parts to find the reduction formula. Let $F(x) = x^n, G'(x) = e^{2x}$. [1 point for using integration by parts, 1 point for choice of $F(x), G'(x)$]

Then $F'(x) = nx^{n-1}, G(x) = \frac{1}{2}e^{2x}$, so we get:

$$\int x^n e^{2x} dx = \frac{1}{2}x^n e^{2x} - \int \frac{n}{2}x^{n-1}e^{2x} dx = \frac{1}{2}x^n e^{2x} - \frac{n}{2}\int x^{n-1}e^{2x} dx$$

[1 point for correctly finding $F'(x), G(x)$, 1 point for integrating correctly, and then 1 point for pulling out the constant $\frac{n}{2}$ so that you actually get a reduction formula]

Note: You could do this problem with $F(x) = e^{2x}, G'(X) = x^n$, which we gave points for if done correctly.

(b) (5 points) Notation: Let $I_n = \int x^n e^{2x} dx$. The reduction formula is $I_n = \frac{1}{2}x^n e^{2x} - \frac{n}{2}I_{n-1}$.

We first calculate I_0, which is just $\int e^{2x} dx$. This can be done directly, and integrates to $\frac{1}{2}e^{2x} + C$. Using the reduction formula, we can then plug I_0 in to the right hand side of $I_1 = \frac{1}{2}xe^{2x} - \frac{1}{2}I_0$ to find I_1. We then plug I_1 in to the reduction formula when $n = 2$ to find I_2, and then plug I_2 in to the reduction formula for $n = 3$, etc. We do this process of finding I_n by multiplying I_{n-1} by $\frac{n}{2}$ and then subtracting this from $\frac{1}{2}x^n e^{2x}$ a total of 50 times until we reach I_{50}. This is the integral we want, so we are done.

Note: You could alternately start at I_{50} and work your way down to I_0 by repeated substitution.

Things that were needed to get full points:

- Starting at I_0 and working up or starting at I_{50} and working down.
- Mentioning that I_0 could be integrated directly.
- Describing repeated use of the reduction formula.
- Mentioning explicitly or implicitly that this would occur a total of 50 times.
- Some description of the algebra required to do this repeated substitution.

(c) (5 points) Instead of using the reduction formula, we could have used integration by parts on the integral $\int x^{50}e^{2x} dx$. With the correct choice of $F(x), G'(x)$, this would lead to an expression in terms of the integral $\int x^{49}e^{2x} dx$. We could then apply integration by parts again to get an expression in terms of $\int x^{48}e^{2x} dx$, and continue using integration by parts 50 times in this manner. The expression would then be in terms of $\int e^{2x} dx$, which can be integrated directly as $\frac{1}{2}e^{2x} + C$.

Note: You could alternately start at $\int e^{2x} dx$ and use a different choice of $F(x), G'(x)$ to eventually get to $\int x^{50}e^{2x} dx$.

Things that were needed for full points:

- Mentioning you could use integration by parts.

- Stating that you use integration by parts on $\int x^{50}e^{2x}$ in order to reduce the exponent of x.
- Describing the repeated use of integration by parts.
- Stating that the exponent of x decreases by 1 at each step or that you would integrate by parts 50 times.
- Mentioning that the base case, $\int e^{2x}dx$, can be integrated directly.

Problem 5: (15 points)

The problem asks to find $\int \frac{x^4+2x^3+3x^2+3x+2}{(x+1)(x^2+x+1)}dx$. We are going to need to do partial fraction decomposition, but since the degree of the numerator is 4 and the degree of the denominator is 3, we first must do polynomial long division. Keep in mind that polynomial long division only works with simplified expressions, so we have to multiply out the denominator and divide $\frac{x^4+2x^3+3x^2+3x+2}{x^3+2x^2+2x+1}$.

After long division, our new integral is $\int x+\frac{x^2+2x+2}{x^3+2x^2+2x+1}dx$. To do partial fraction decomposition on the second term, we need the denominator to be factored. We know from the original statement of the problem that it factors to $(x+1)(x^2+x+1)$. So we will perform partial fractions on $\frac{x^2+2x+2}{(x+1)(x^2+x+1)}$, which has the form

$$\frac{x^2+2x+2}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}.$$

Multiplying by $(x+1)(x^2+x+1)$ on both sides, we get

$$\begin{aligned} x^2+2x+2 &= A(x^2+x+1) + (Bx+C)(x+1) \\ &= Ax^2 + Ax + A + Bx^2 + Bx + Cx + C \\ &= Ax^2 + Bx^2 + Ax + Bx + Cx + A + C \\ &= (A+B)x^2 + (A+B+C)x + (A+C). \end{aligned}$$

Equating coefficients, we get

$$A+B=1, \qquad A+B+C=2, \qquad A+C=2.$$

The solution to this system of equations is $A=1$, $B=0$, and $C=1$. Thus we can rewrite the original integral as

$$\int x + \frac{1}{x+1} + \frac{1}{x^2+x+1}dx.$$

Getting to this point in the problem was worth 6 points: 2 points for correct long division, 3 points for correctly finding A, B, C, and 1 point for rewriting the integral in correct form.

We will do each integral individually. We have

$$\int x dx = \frac{x^2}{2} + C.$$

For $\int \frac{1}{x+1}dx$, we can make the substitution $u = x+1$, and so $du = dx$, thus we have

$$\int \frac{1}{x+1}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|x+1| + C.$$

Correctly integrating these first two integrals was worth 3 points. Correctly integrating just one of these integrals was worth 1 point.

For the last integral, we must complete the square on the denominator and put it in the form $\int \frac{1}{v^2+1}dv$. We have

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4},$$

and so

$$\begin{aligned}\frac{1}{x^2+x+1} &= \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} \\ &= \frac{1}{\frac{3}{4}\left[\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}}+1\right]} \\ &= \frac{4}{3}\cdot\frac{1}{\frac{4\left(x+\frac{1}{2}\right)^2}{3}+1} \\ &= \frac{4}{3}\cdot\frac{1}{\left(\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right)^2+1} \\ &= \frac{4}{3}\cdot\frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}.\end{aligned}$$

To integrate, we will make the substitution $v = \frac{2x+1}{\sqrt{3}}$, and so $dv = \frac{2}{\sqrt{3}}dx$, and thus $dx = \frac{\sqrt{3}}{2}dv$.

We have

$$\begin{aligned}\int \frac{1}{x^2+x+1}dx = \frac{4}{3}\int \frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}dx &= \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int\frac{1}{v^2+1}dv \\ &= \frac{2}{\sqrt{3}}\arctan v + C = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C.\end{aligned}$$

Correctly integrating this final integral was worth 6 points: 2 points for correctly completing the square, 2 points for correctly writing in in the form $\frac{1}{v^2+1}$, and 2 points for correctly integrating this to $\arctan v$.

Thus the final answer is

$$\int \frac{x^4+2x^3+3x^2+3x+2}{(x+1)(x^2+x+1)}dx = \frac{x^2}{2} + \ln|x+1| + \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C.$$

Problem 6:

(a) (3 points) It is possible to find $\int x^2 dx$ without knowing the Fundamental Theorem of Calculus. By definition, the indefinite integral $\int x^2 dx$ is any anti-derivative of x^2. Indefinite integrals only require anti-differentiation.

(b) (7 points) The Fundamental Theorem of Calculus tells us that definite integrals of continuous functions can be computed using indefinite integrals. Specifically, it says:

If f is a continuous function and F is an anti-derivative of f, then $\int_a^b f(x)dx = F(b) - F(a)$ for all real numbers a and b.

The Fundamental Theorem of Calculus does not help us with $\int x^2 dx$, since it does not give us any way to compute indefinite integrals. The Fundamental Theorem of Calculus doesn't tell us how to find anti-derivatives, it just gives us something to do with them once we've found them.

The Fundamental Theorem of Calculus does help us with $\int_0^1 x^2 dx$, because it tells us how to compute definite integrals of continuous functions. Specifically, since we know that x^2 is a continuous function and $\left(\frac{x^3}{3}\right)' = x^2$, the Fundamental Theorem of Calculus tells us that

$$\int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \left(\frac{(1)^3}{3} - \frac{(0)^3}{3}\right) = 1/3.$$

Makeup Problem 7 (10 points) When we write $\int x^2 dx = \frac{x^3}{3} + C$ we mean that for any constant C, the derivative of $\frac{x^3}{3} + C$ is equal to x^2, and that every anti-derivative of x^2 is equal to $\frac{x^3}{3} + C$ for some choice of a constant C.

Since the derivative of a constant is 0, adding a constant to any anti-derivative will give us another anti-derivative of the same function. Also, since the only functions whose derivatives are 0 are constant functions, any two anti-derivatives of the same function differ by a constant.

In general, when we add $+C$ to our indefinite integrals, we are acknowledging that adding on any constant will give us another anti-derivative of the same function, and that any other anti-derivative of the same function can be found by adding on a certain constant.

Problem 7:

(a) (13 points) This problem needed to be broken down in to cases, depending on what p is.

Case 1: $p \geq 0$. In this case, $\int_0^1 x^p dx$ is a proper integral, so this integral has to exist. Explicitly:

$$\int_0^1 x^p dx = \frac{x^{p+1}}{p+1}\bigg|_0^1 = \frac{1}{p+1}$$

Case 2: $p < 0$. In this case, x^p is not defined at $x = 0$, so we need to set this up as an improper integral. We get that this equals:

$$\lim_{a \to 0} \int_a^1 x^p dx$$

[2 points for correctly writing the improper integral for $p < 0$]

How we do this integral depends on whether $p = -1$ or not. [1 points for splitting this up in to the cases where $p \neq -1$ and where $p = -1$]

As long as $p \neq -1$, this equals:

$$\lim_{a \to 0} \left[\frac{x^{p+1}}{p+1}\right]_a^1 = \lim_{a \to 0} \frac{1}{p+1} - \frac{a^{p+1}}{p+1}$$

[4 points for writing this limit correctly]

If $-1 < p < 0$, this limit exists and equals $\frac{1}{p+1}$. If $p < -1$, this integral does not exist as a^{p+1} goes to ∞ as $a \to 0$. [2 points for doing this limit calculation]

Finally, if $p = -1$, then we have:

$$\begin{aligned}\lim_{a \to 0} \int x^{-1} dx &= \lim_{a \to 0} (\ln(1) - \ln(a)) \\ &= \lim_{a \to 0} (-\ln(a)) = \infty\end{aligned}$$

So, this integral does not exist for $p = -1$. [2 points for working out this case correctly]

Putting this all together, we get that the integral exactly when $p > -1$. [2 points for the correct final answer]

(b) (2 points) You needed to state that this integral exists when $p < -1$ and does not exist for $p \geq -1$.

Problem 8:

(a) (5pts) Definition: $\lim_{a\to\infty}\int_1^a \frac{dx}{(x^4+2x^2-x+1)^{1/4}}$
No Credit for "area under curve".

(b) (15pts) When $x \geq 1$, we have $x^2 \leq x^4$ and $-x+1 \leq 0$, so $x^4+2x^2-x+1 \leq 3x^4$ and

$$\frac{1}{(x^4+2x^2-x+1)^{1/4}} \geq \frac{1}{(3x^4)^{1/4}} = \frac{1}{(3)^{1/4}x}.$$

By comparison theorem,

$$\begin{aligned} \int_1^\infty \frac{1}{(x^4+2x^2-x^1)^{1/4}}dx &\geq \int_1^\infty \frac{1}{3^{1/4}x}dx \\ &= \frac{1}{3^{1/4}} \lim_{a\to\infty} \int_1^a \frac{dx}{x} \\ &= \frac{1}{3^{1/4}} \lim_{a\to\infty} \ln a \text{ (which does not exist, } \infty) \end{aligned}$$

So the original integral $\int_1^\infty \frac{dx}{(x^4+2x^2-x+1)^{1/4}}$ does not exist.
Common Mistakes:

- 6pts for the estimation $\frac{1}{(x^4+2x^2-x+1)^{1/4}} \sim \frac{1}{x}$ and stating that $\int_1^\infty \frac{dx}{x}$ DNE, without actually unsing comparison theorem.
- 3pts for wrong comparison theorem, such as
  1. $\frac{1}{(x^4+2x^2-x+1)^{1/4}} \leq g(x)$, and $\int_1^\infty g(x)dx$ exists;
  2. $\frac{1}{(x^4+2x^2-x+1)^{1/4}} \leq \frac{1}{x}$ and $\int_1^\infty \frac{1}{x}dx$ DNE, so the $\int_1^\infty \frac{dx}{(x^4+2x^2-x+1)^{1/4}}$ DNE.
- 10pts for students who use the right comparison theorem with wrong inequality, like $\frac{1}{(x^4+2x^2-x+1)^{1/4}} \geq \frac{1}{x}$.

Full credits for the correct inequality(with the proof) and correct comparison theorem