Math 863 Notes
Algebraic geometry II

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Contents

1 2015-01-21: Sheaves

1.1 Course outline

Rough outline:

- Sheaves ("geometric")
- Schemes ("algebraic")
- Moduli spaces
- Other?

There will be weekly homework and no exams.
References:

- Hartshorne, *Algebraic Geometry*
- Shafarevich, *Basic Algebraic Geometry*
- Ravi Vakil's notes

1.2 Sheaves

Geometry studies "spaces". The difference between different flavors of geometry is which kinds of functions on spaces are considered.

Examples:

- Topological spaces with continuous functions
- Differentiable manifolds with differentiable functions
- Complex manifolds with complex analytic functions

- Algebraic varieties with regular functions

Problem: Sometimes, there are too few globally defined functions. For example, on a compact complex manifold or a projective algebraic variety, the only globally defined functions are constants.

Instead, we want to study locally defined functions, leading to a new structure: to each open subset $U \subset X$, we assign the space of functions on U.

Definition 1.1 (Presheaf)**.** Let X be a topological space. A *presheaf* F on X is the following data:

(1) a set $F(U)$ for each open $U \subset X$, whose elements are called *sections* of F over U;

(2) maps $f \mapsto f|_V : F(U) \to F(V)$ for each inclusion $V \subset U$ such that $F(U) \to F(U)$ is the identity and $(f|_V)|_W = f|_W$ for any $W \subset U \subset V$.

In other words, if $\mathrm{Open}(X)$ is the poset of open subsets of X, then a presheaf on X is a functor $F : \mathrm{Open}(X)^{op} \to \mathbf{Set}$. A morphism of presheaves $F \to G$ is a natural transformation.

Remark 1.2. More generally, if C is any category, a C-presheaf is a functor $\mathrm{Open}(X)^{op} \to C$.

Example 1.3 (Constant presheaf). Fix a set S. Define $F(U) = S$ for all open U, and let all restriction maps be the identity on S.

Definition 1.4 (Sheaf)**.** A *sheaf* F is a presheaf such that, for every open cover $U = \bigcup_i U_i$:

(1) If $f, g \in F(U)$ such that $f|_{U_i} = g|_{U_i}$ for all i, then $f = g$. (If F satisfies this condition, we say F is a *separated presheaf*.)

(2) Given $f_i \in F(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, there exists $f \in F(U)$ such that $f|_{U_i} = f_i$ for all i. By (1), such an f is unique.

In other words, we have an equalizer diagram

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j),$$

where the morphisms are $f \mapsto (f|_{U_i})$, $(f_i) \mapsto (f_i|_{U_i \cap U_j})$, and $(f_j) \mapsto (f_j|_{U_i \cap U_j})$. A morphism of sheaves $F \to G$ is a morphism of presheaves where the source and target are sheaves.

Remark 1.5. Similarly, we define C-sheaves for any *complete* category C, i.e., C has all small limits (equivalently, C has small products and equalizers).

Example 1.6 ("Nice" functions). Let X be a "space" (topological space, manifold, variety, etc.), and let F be a sheaf of "nice functions" (continuous, smooth, analytic, regular, etc.) on X.

Example 1.7 (Bounded continuous functions). Let X be a topological space, and let $F(U)$ be the set of bounded continuous functions $f : U \to \mathbb{R}$. This is a separated presheaf, but not a sheaf — a locally bounded function is not necessarily bounded!

Example 1.8 (Sigular cochains). Singular chains form a co-presheaf (a *covariant* functor $\mathrm{Open}(X) \to \mathbf{Set}$), but not a presheaf. Singular cochains, on the other hand, form a presheaf. However, this is not even a separated presheaf, since there can be simplices on U that aren't wholly contained in any of the U_i.

Example 1.9 (Locally constant sheaf). Fix a set S. For each open $U \subset X$, let $F(U)$ be the set of locally constant maps $U \to S$. This is a sheaf.

2 2015-01-23: Germs and stalks

2.1 More examples

Example 2.1. Let X and Y be topological spaces. Then we can define a sheaf F on X by $F(U) = \mathrm{Hom}_{\mathbf{Top}}(U, Y)$.

Example 2.2. Given $\pi : E \to X$, consider the *sheaf of sections* of π:

$$F(U) = \left\{ s : U \to E \text{ continuous} \;\middle|\; \pi \circ s = \mathrm{id} \right\}.$$

Example 2.3. Let X be a differentiable manifold. Then $U \mapsto C^\infty(U)$ defines a sheaf of $\mathbb{R}$-algebras on X. Also, for $k \geq 1$, $U \mapsto \Omega^k(U)$ is a sheaf of smooth differential k-forms on X, which is a sheaf of $\mathbb{R}$-vector spaces. We obtain a complex of sheaves of $\mathbb{R}$-vector spaces

$$C^\infty \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \dots$$

2.2 Germs and stalks

Definition 2.4. Let F be a presheaf on X, and fix a point $x \in X$. The *stalk* of F at x is $F_x := \mathrm{colim}_{U \ni x} F(U)$. An element of F_x is called a *germ of a section* of F at x.

More explicitly, an element of F_x is represented by a pair (U, s), where U is an open neighborhood of x and $s \in F(U)$. Pairs (U, s) and (V, t) are considered equivalent iff there is an open $W \subset U \cap V$ such that $s|_W = t|_W$.

Example 2.5. Let X be a (real or complex) analytic manifold, and let F be the sheaf of analytic functions on X. For all $x \in X$, the stalk F_x is the ring of power series convergent on some neighborhood of x.

Example 2.6. Let X be an algebraic variety, and let $\mathcal{O}_X$ be the sheaf of regular functions on X. The stalk $\mathcal{O}_{X,x}$ is the localization.

Remark 2.7. If C is an arbitrary complete category, then C-sheaves might not have stalks — we also need C to have filtered colimits. However, stalks will at least exist as *ind-objects*: formal filtered colimits of objects of C.

3 2015-01-26: Sheafification

Definition 3.1. Let F be a presheaf on a space X. Its *sheafification* $\tilde{F}$ is a sheaf on X together with a map $F \to \tilde{F}$ such that for any sheaf G, any morphism $F \to G$ factors uniquely as $F \to \tilde{F} \to G$.

Remark 3.2. Let $\mathrm{PShf}(X)$ (resp. $\mathrm{Shf}(X)$) be the category of presheaves (resp. sheaves) on X. Then sheafification is the left adjoint to the fully faithful inclusion $\mathrm{PShf}(X) \to \mathrm{Shf}(X)$.

We can construct $\tilde{F}$ as follows:

$$\tilde{F}(U) = \operatorname*{colim}_{U = \bigcup_i U_i} \left\{ (s_i \in U_i) : s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j} \ \forall i, j \right\},$$

where the colimit is with respect to refining covers.

Example 3.3. Let F be the constant presheaf on S. Then $\tilde{F}$ is the constant sheaf on S, i.e., the sheaf of locally constant maps $U \to S$. (In particular, $F(\varnothing) = \{*\}$.)

Proposition 3.4. *Let F be a presheaf, and fix $x \in X$. The sheafification map $F \to \tilde{F}$ induces an isomorphism $F_x \xrightarrow{\simeq} \tilde{F}_x$.*

Proposition 3.5. *If $\varphi : F \to G$ is a morphism of sheaves such that $\varphi_x : F_x \to G_x$ is an isomorphism for all $x \in X$, then φ is an isomorphism.*

Corollary 3.6. *If F is a presheaf and F' is a sheaf on X together with $\varphi : F \to F'$ which induces a bijection on all stalks, then F' is the sheafification of F.*

4 2015-01-28: Maps between sheaves

Exercise 4.1. If $\varphi : F \to G$ is a homomorphism of sheaves of groups, then the sub-presheaf $\ker(\varphi) = \{s \in F \mid \varphi(s) = e\}$ is a sheaf.

However, the same is not true for images.

Example 4.2. Let $\mathcal{O}$ be the sheaf of $\mathbb{C}$-valued functions on X, and let $\mathcal{O}^*$ be the sheaf of $\mathbb{C}^\times$-valued functions on X. Then $\exp : \mathcal{O} \to \mathcal{O}^*$ is a homomorphism of sheaves of abelian groups. The kernel $\ker(\exp)$ is the (locally) constant sheaf associated to the additive group $2\pi i \cdot \mathbb{Z}$.

Since we can take logarithms locally but not globally, the maps on sections $\mathcal{O}(U) \to \mathcal{O}^*(U)$ are locally surjective, but not surjective. Hence, the *image presheaf*, defined by $U \mapsto \operatorname{im}(\mathcal{O}(U) \to \mathcal{O}^*(U))$, is not a sheaf.

Definition 4.3. Let $\varphi : F \to G$ be a morphism of sheaves. The *image sheaf* $\operatorname{im}(\varphi)$ is the sheafification of the image presheaf $U \mapsto \operatorname{im}(F(U) \to G(U))$.

The sheafification of $\exp : \mathcal{O} \to \mathcal{O}^*$ is all of $\mathcal{O}^*$, so exp is an epimorphism of sheaves (but not an epimorphism of presheaves). Even though $\mathrm{Shf}(X)$ is a full subcategory of $\mathrm{PShf}(X)$, these categories have different notions of epimorphism.

5 2015-01-30: Espace étalé of a sheaf

Presheaves of abelian groups on a space X form an abelian category; sums of morphisms, direct sums, kernels, and cokernels are taken over each open subset U. The functor $F \mapsto F(U) : \mathrm{PShf}_{\mathbf{Ab}}(X) \to \mathbf{Ab}$ is exact.

Sheaves of abelian groups on a space X also form an abelian category, but cokernels require sheafification. We have an adjunction between sheafification $F \mapsto \tilde{F} : \mathrm{PShf}_{\mathbf{Ab}}(X) \to \mathrm{Shf}_{\mathbf{Ab}}(X)$ and inclusion $\mathrm{Shf}_{\mathbf{Ab}}(X) \hookrightarrow \mathrm{PShf}_{\mathbf{Ab}}(X)$; sheafification is exact, and inclusion is full and left exact (but not right exact). Also, for each $x \in X$, the functor $F \mapsto F_x : \mathrm{Shf}_{\mathbf{Ab}}(X) \to \mathbf{Ab}$ is exact.

Example 5.1. The exact sequence $0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 1$ induces an exact sequence on stalks, but not always on open subsets.

5.1 Another approach to sheaves

Let F be a sheaf on a space X. We proved that $s \in F(U)$ is determined by the germs $s_x \in F_x$ for all $x \in U$. Put $\mathcal{E}(F) = \bigsqcup_x F_x$, and define $\pi : \mathcal{E}(F) \to X$ by $\pi(F_x) = x$. Each $s \in F(U)$ is a section $s : U \to \mathcal{E}(F)$ such that $\pi \circ s = \mathrm{id}$.

Remark 5.2. There is a natural injection $F \hookrightarrow \{\text{sheaf of sections of } \pi\}$.

Definition 5.3. A section $s : U \to \mathcal{E}(F)$ is *representable* if locally over U it comes from a section of F, i.e., for every $x \in U$, there is $x \in V \subset U$ and a section $t \in F(V)$ such that $s(y) = t_y \in F_y$ for all $y \in V$.

Claim 5.4. *F is isomorphic to the sheaf of representable sections of π.*

Let us equip $\mathcal{E}(F)$ with the smallest topology for which all representable sections are continuous.

Lemma 5.5. *The continuous sections of π are exactly the representable sections.*

Proof. For two sections $s, t \in F(U)$, the locus $\{x \in U \mid s_x = t_x\}$ is open in U. □

Each $s \in F(U)$ is a section $s : U \to \mathcal{E}(F)$ such that $\pi \circ s = \mathrm{id}$. Hence, F is isomorphic to the sheaf of continuous sections of π. We call $\mathcal{E}(F)$ the *espace étalé* of F.

The map $\pi : \mathcal{E}(F) \to X$ is a local homeomorphism: For any $y \in \mathcal{E}(F)$, there are neighborhoods $y \in V \subset \mathcal{E}(F)$ and $\pi(y) \in U \subset X$ such that $\pi|_V : V \to U$ is a homeomorphism. Conversely, if $\pi : Y \to X$ is a continuous local homeomorphism, then Y is the espace étalé of the sheaf of continuous sections of π.

6 2015-02-02: Algebraic varieties

6.1 Sheaves of sections

Continuing from last time, the espace étalé construction gives an equivalence

$$\mathrm{Shf}(X) \longleftrightarrow \{\text{maps } \pi : Y \to X \text{ such that } Y \text{ is locally homeomorphic to } X\}.$$

The inverse of the functor $F \mapsto \mathcal{E}(F)$ is the functor sending Y to its sheaf of sections.

As an application, given a presheaf F, we can construct $\mathcal{E}(F)$ in the same way, and the sheafification $\tilde{F}$ is the sheaf of sections of $\mathcal{E}(F)$. Explicitly,

$$\tilde{F}(U) = \left\{(s_x \in F_x)_{x \in U} \;\middle|\; \forall x \in U,\ \exists V \ni x \text{ and } t \in F(V) \text{ such that } s_y = t_y\ \forall y \in V\right\}.$$

Example 6.1 (Constant sheaf). Fix S and consider the constant presheaf F. For all $x \in X$, $F_x = S$, so $\mathcal{E}(F) = X \times S \to X$, where $X \times S$ has the product topology (S is discrete). Then $\tilde{F}(U)$ is the set of locally constant maps $U \to S$.

Example 6.2 (Skyscraper sheaf). Fix S and $x \in X$. Define $F(U) = S$ if $x \in U$ and $F(U) = \{*\}$ if $x \notin U$. (Exercise: This is a sheaf.) Then for $y \in X$,

$$F_y = \begin{cases} S & \text{if } y \in \overline{\{x\}}, \\ \{*\} & \text{if } y \notin \overline{\{x\}}. \end{cases}$$

This is called a *skyscraper sheaf*.

6.2 Algebraic varieties

Let X be an affine algebraic variety over an algebraically closed field k. Consider X as an abstract variety: X is a set together with a class of functions $k[X] \subset \{f : X \to k\}$ such that there exists a bijection between X and an algebraic subset of $\mathbb{A}^n$ under which $k[X]$ become polynomial functions.

One can also start with a finitely-generated reduced k-algebra $R = k[X]$, then put $X = \mathrm{mSpec}(R)$. We can then view $f \in R$ as a function on X: for a point $x \in X$ corresponding to the maximal ideal $\mathfrak{m}_x \subset R$, define $f(x) = f + \mathfrak{m}_x \in R/\mathfrak{m}_x \cong k$.

Consider X with the Zariski topology. Define the *sheaf of regular functions* $\mathcal{O}_X$ by

$$\mathcal{O}_X(U) = \left\{ f : U \to k \;\middle|\; f \text{ is regular} \right\}.$$

It is a nontrivial statement that $\mathcal{O}_X(X) = k[X]$.

7 2015-02-04: Sheaves of regular functions

Open sets of the form $D(g) = X - Z(g)$ form a basis for the Zariski topology on X. The coordinate ring $R = k[X]$ is a finitely-generated reduced k-algebra.

Put $\mathcal{O}_X(D(g)) = R[g^{-1}]$ for $g \in R$. If $D(g_2) \supset D(g_1)$, then $g_1 \mid g_2^N$ for some $N \in \mathbb{N}$, so there is a localization map $\mathcal{O}_X(D(g_2)) = R[g_2^{-1}] \to R[g_1^{-1}] = \mathcal{O}_X(D(g_1))$. For any open $U \subset X$, define $\mathcal{O}_X(U) = \lim_{D(g) \subset U} \mathcal{O}_X(D(g))$. This makes $\mathcal{O}_X$ into a presheaf.

Proposition 7.1. *Let S be a commutative ring. Fix elements $g_i \in S$ such that $(g_i)_i = (1)$. Given $\varphi_i \in S[g_i^{-1}]$ such that $\varphi_i = \varphi_j$ in $S[(g_i g_j)^{-1}]$ for all i, j, there is a unique $f \in S$ such that $f = \varphi_i$ in $S[g_i^{-1}]$ for all i.*

Proof. Write $\varphi_i = f_i / g_i^{\kappa_i}$. Then $f_i / g_i^{\kappa_i} = f_j / g_j^{\kappa_j}$ in $S[(g_i g_j)^{-1}]$ means $(g_i g_j)^{m_{ij}} (f_i g_j^{\kappa_j} - f_j g_i^{\kappa_i}) = 0$ in S. Consider a finite subfamily such that $(g_1, \ldots, g_n) = (1)$. Replace g_i by g_i^N for $N \gg 0$ so that $\varphi_i = f_i / g_i$ and $f_i g_j - f_j g_i = 0$ for all i, j. Write $1 = \sum_i g_i h_i$. Now take $f = \sum_i f_i h_i$. Then $f g_j = \sum_i f_i g_j h_i = \sum_i f_j g_i h_i = f_j$, so $f = f_j / g_j = \varphi_j$ in $S[g_j^{-1}]$.

For uniqueness, suppose $f = \varphi_j = f_j / g_j$ in $S[g_j^{-1}]$. Then $g_j^N f g_j = g_j^N f_j$. Replace g_j by g_j^M, so that $f g_j = f_j$. Hence, $f = \sum_j f g_j h_j = \sum_j f_j h_j$. $\square$

So $\mathcal{O}_X$ is a sheaf of k-algebras on X such that $\mathcal{O}_X(D(g)) = k[X][g^{-1}]$ for all $g \in k[X]$. This is like partitions of unity in differential geometry.

8 2015-02-06: Ringed spaces

Definition 8.1. A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$, the *structure sheaf* of X. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ together with a sheaf morphism $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$. More explicitly, for every open subset $V \subset Y$, $f^\sharp(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ is a ring morphism, and these morphisms are compatible with restriction.

Example 8.2. Given topological spaces (smooth manifolds, complex manifolds, algebraic varieties) X and Y, a continuous (smooth, holomorphic, regular) map $X \to Y$ gives a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. These give functors to ringed spaces; we do not claim these functors are fully faithful.

Given affine algebraic varieties X and Y over a field k, when do morphisms $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ come from morphisms of algebraic varieties?

Example 8.3. A morphism of ringed spaces $(\operatorname{Spec} k, k) \to (\operatorname{Spec} k, k)$ is the same thing as a ring endomorphism of k. This isn't what we want.

So, instead, consider k-ringed spaces, i.e., spaces equipped with sheaves of k-algebras. The pullback of functions for a morphism of k-ringed spaces is required to be a morphism of k-algebras. There is a faithful, but not full, functor from k-ringed spaces to ringed spaces.

The next step is to prove that the category of affine algebraic varieties over k embeds fully faithfully into the category of k-ringed spaces. We'll do this next time.

9 2015-02-09: Affine varieties as ringed spaces

Here's what we'll talk about next in the course:

(1) Algebraic varieties via sheaves (right now)

(2) Schemes (next week; Daniel Erman will be substituting)

(3) Quasicoherent sheaves and vector bundles

(4) Cohomology

Theorem 9.1. *Let X and Y be affine algebraic varieties over k. Morphisms of k-ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are in natural bijection with regular maps $X \to Y$.*

Proof. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of k-ringed spaces. Then the pullback $\varphi = f^\sharp(Y) : k[Y] = \mathcal{O}_Y(Y) \to \mathcal{O}_X(f^{-1}(Y)) = \mathcal{O}_X(X) = k[X]$ induces a regular map $X \to Y$. The nontrivial part is to reconstruct f from φ.

For $x \in X$, put $\mathfrak{m}_x = \{f \in k[X] : f(x) = 0\} \subset k[X]$, and likewise for $y \in Y$. Then $\varphi^{-1}(\mathfrak{m}_x) \subset k[Y]$ is a maximal ideal. We claim that $\mathfrak{m}_y = \varphi^{-1}(\mathfrak{m}_x)$, where $y = f(x)$.

Given $g \in k[Y]$ such that $\varphi(g)(x) = 0$, we have $g(y) = 0$. Thus, $\varphi^{-1}(\mathfrak{m}_y) \subseteq \mathfrak{m}_y$. Indeed, if $g(y) \neq 0$, then the germ of g is invertible in $\mathcal{O}_{Y,y}$, so the germ of $\varphi(g)$ is invertible in $\mathcal{O}_{X,x}$, whence $\varphi(g)(x) \neq 0$. Since $\varphi^{-1}(\mathfrak{m}_x)$ is maximal, $\varphi^{-1}(\mathfrak{m}_x) = \mathfrak{m}_y$, as claimed.

Next, we must show that f is equal to the morphism $\tilde{f} : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ induced by φ. We know the pullbacks agree on $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ and that $f = \tilde{f}$ as continuous maps. It remains to show that for any principal open $V = D(g) \subset Y$ (with $g \in k[Y]$),

$$f^\sharp(V) = \tilde{f}^\sharp(V) : k[Y][g^{-1}] = \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V)) = \mathcal{O}_X(\tilde{f}^{-1}(V)).$$

Since $f^\sharp(V)|_{k[Y]} = \tilde{f}^\sharp(V)|_{k[Y]}$, this follows from general properties of localization. $\square$

10 2015-02-11: Abstract algebraic varieties

Let F be a sheaf on a space X. For any open $U \subset X$ define the *restriction* $F|_U$ by $F|_U(V) = F(V)$ for any open $V \subset U$. This is also a sheaf. If F has additional structure, so does $F|_U$.

In particular, $F \mapsto F|_U : \mathrm{Shf}_{\mathbf{Ab}}(X) \to \mathrm{Shf}_{\mathbf{Ab}}(U)$ is an exact functor. The key fact is that $F_x = (F|_U)_x$ for all $x \in U$.

Fact 10.1. If X is a (k-)ringed space, then any open subset $U \subset X$ is a (k-)ringed space $(U, \mathcal{O}_X|_U = \mathcal{O}_U)$. There is a natural morphism of (k-)ringed spaces $(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$.

Definition 10.2. An *abstract algebraic variety* over k is a k-ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to an affine algebraic variety, i.e., there is an open covering $X = \bigcup U_i$ such that $(U_i, \mathcal{O}_X|_{U_i}) \cong (V_i, \mathcal{O}_{V_i})$ for some affine algebraic varieties V_i.

Conversely, given a topological space $X = \bigcup U_i$ and homeomorphisms $\varphi_i : U_i \to V_i$ such that $\varphi_j \circ \varphi_i^{-1}$ are regular, X has a natural structure of an algebraic variety.

Exercise 10.3. Let X and Y be k-ringed spaces. Consider the presheaf on X defined by

$$U \mapsto \{\text{morphisms of } k\text{-ringed spaces } (U, \mathcal{O}_U) \to (Y, \mathcal{O}_Y)\}.$$

Prove this is a sheaf.

Remark 10.4. Morphisms of algebraic varieties are morphisms between them as k-ringed spaces.

Remark 10.5 (Differential analogy). A *differentiable manifold* is an $\mathbb{R}$-ringed space that is locally isomorphic to $(\mathbb{R}^n, \mathcal{C}^\infty)$, where $\mathcal{C}^\infty$ is the sheaf of C^∞-functions. (Usually, one also requires the space to be second-countable and Hausdorff.)

Remark 10.6. Similarly, algebraic varieties are often required to be compact and separated.

Example 10.7. Here are some examples of (abstract) algebraic varieties.

- Quasi-projective varieties: locally closed subsets of $\mathbb{P}^n$ for some n.
- Arbitrary disjoint unions of quasi-projective varieties.
- Two copies of $\mathbb{P}^1$, glued away from one point. This is non-separated.
- A $\mathbb{Z}$-indexed sequence of projective lines, with the n-th line glued to the $(n-1)$-st and $(n+1)$-st line at a single point.

11 2015-02-13: Operations on sheaves

Fix a continuous map $f : X \to Y$.

11.1 Direct image

Definition 11.1 (Direct image)**.** Given a presheaf F on X, the *direct image* (or *pushforward*) f_*F is the presheaf on Y defined by $f_*F(V) = F(f^{-1}(V))$ for all open $V \subset Y$.

Exercise 11.2 (Easy exercise). If F is a sheaf, then f_*F is also a sheaf.

Remark 11.3. Classically, given a sheaf F on X and a sheaf G on Y, a morphism $G \to f_*F$ is called a *cohomomorphism*.

Example 11.4. Let F be a sheaf on X, and let $p : X \to \{*\}$ be the unique map. Then $p_*F = F(X)$.

Example 11.5. Let $i : \{*\} \to X$ be a map. Let S be a set, viewed as a sheaf on $\{*\}$. Then i_*S is the skyscraper sheaf of S at $i(*) \in X$.

Proposition 11.6 (Properties of direct image)**.** *(1) f_* is a functor.*

(2) If $g : X \to Y$ and $f : Y \to Z$ are two continuous maps, then $(fg)_* = g_*f_*$.

(3) f_* is an additive, left exact functor on sheaves of abelian groups.

Example 11.7. Let X be an algebraic variety. Let $j : U \hookrightarrow X$ be an open embedding. Let F be a sheaf on U. Then $(j_*F)|_U = F$. In particular, $(j_*F)_x = F_x$ for each $x \in j(U)$.

For $x \in X - j(U)$, the stalk $(j_*F)_x$ is more complicated in general. For example, if $X = \mathbb{A}^1$ and $U = \mathbb{A}^1 - \{0\}$, then $(j_*\mathcal{O}_U)_0$ is the ring of germs of *rational* functions at 0. If $X = \mathbb{A}^2$ and $U = \mathbb{A}^2 - \{0\}$, then $(j_*\mathcal{O}_U)_0$ is the ring of germs of *regular* functions at 0.

11.2 Inverse image

Definition 11.8. Given a presheaf G on Y, the *inverse image* (or *pullback*) $f^{-1}G$ is the presheaf on X defined by $(f^{-1}_{\text{pre}}G)(U) = \operatorname{colim}_{V \supset f(U)} G(V)$. If G is a sheaf, define $f^{-1}G$ to be the sheafification of $f^{-1}_{\text{pre}}G$.

Proposition 11.9. *For each $x \in X$, $(f^{-1}G)_x = G_{f(x)}$.*

Hence, we can equivalently define $f^{-1}G$ be its espace étalé: $\mathcal{E}(f^{-1}G) = \mathcal{E}(G) \times_Y X$.

Exercise 11.10. Let $p : \mathcal{E}(G) \to Y$ be the natural map. Then

$$(f^{-1}G)(U) = \{s : U \to \mathcal{E}(G) : p \circ s = f\}.$$

Example 11.11. Let $p : X \to \{*\}$ be the unique map. Let S be a set, considered as a sheaf on $\{*\}$. Then $p^{-1}S = \underline{S}$ is the constant sheaf on F associated to S.

Example 11.12. Let $i : \{*\} \to X$ be a map. Let F be a sheaf on X. Then $i^{-1}F = F_{i(*)}$.

Proposition 11.13 (Properties of inverse image)**.** *(1) f^{-1} is a functor.*

(2) If $g : X \to Y$ and $f : Y \to Z$ are two continuous maps, then $(fg)^{-1} = f^{-1}g^{-1}$.

(3) f_* is an additive, exact functor on sheaves of abelian groups.

Proposition 11.14. *There are natural bijections*

$$\operatorname{Hom}_{\operatorname{Shf}(X)}(f^{-1}G, F) \longleftrightarrow \operatorname{Hom}_{\operatorname{Shf}(Y)}(G, f_*F) \longleftrightarrow \{\textit{cohomomorphisms } G \rightsquigarrow F\}.$$

In particular, f^{-1} is left adjoint to f_*.

12 2015-02-16: Affine schemes

Guest lecture by Daniel Erman.

In the theory of varieties, affine varieties over an algebraically closed field k correspond to finitely-generated reduced k-algebras. A better theory would include, among other things:

- Solutions of polynomials over $\mathbb{Z}, \mathbb{Q}, \mathbb{Q}_p, F_q$, etc.
- Nonreduced objects (double points, etc.)
- Local neighborhoods of points.

Grothendieck's idea was to replace finitely-generated reduced k-algebras with arbitrary commutative rings. The corresponding geometric objects are called *affine schemes*.

12.1 Affine schemes as sets

Given a commutative ring R, define the *prime spectrum* $\operatorname{Spec} R = \{P \subset R \text{ prime}\}$. We can interpret elements of R as "functions" on $\operatorname{Spec} R$ via the maps $R \to R_P \to \kappa(P) = R_P/P_P$ to the residue fields. Note that these "functions" can take values in different fields.

Example 12.1. If $R = \mathbb{Z}$ and $f = 3 \in \mathbb{Z}$, then $f(p) = \bar{3} \in \mathbb{Z}/(p)$ for each prime number p, and $f(0) = 3 \in \mathbb{Q}$.

12.2 The Zariski topology

The *Zariski topology* on an affine scheme $\operatorname{Spec} R$ is the topology where the closed subsets have the form $V(I) = \{P \supseteq I : P \in \operatorname{Spec} R\}$, where $I \subseteq R$ is an ideal.

Lemma 12.2. *Finite unions and arbitrary intersections of closed subsets are closed.*

Proof. If I and J are ideals of R, then $V(I) \cup V(J) = V(IJ)$. If $\{I_\alpha\}$ is a family of ideals of R, then $\bigcap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$. $\square$

For $f \in R$, the open subset

$$\operatorname{Spec}(R) \setminus V(f) = \{P \in \operatorname{Spec} R : f \notin P\} = \operatorname{Spec}(R[f^{-1}])$$

is called a *basic open affine*. These form a basis for the Zariski topology on $\operatorname{Spec} R$.

Example 12.3. Not every open subset is an affine scheme: the punctured plane $\operatorname{Spec}(\mathbb{C}[x,y]) \setminus V(x,y)$ is not affine.

12.3 Morphisms of affine schemes

A *morphism of affine schemes* $\operatorname{Spec} R \to \operatorname{Spec} S$ is a ring homomorphism $S \to R$.

Example 12.4 (A double point). Morphisms $\operatorname{Spec}(\mathbb{C}[x]/(x)) = \operatorname{Spec}(\mathbb{C}) \to \mathbb{A}^1_{\mathbb{C}}$ correspond to ring maps $\mathbb{C}[y] \to \mathbb{C}$, which are given by $y \mapsto a \in \mathbb{C}$. On the other hand, morphisms $\operatorname{Spec}(\mathbb{C}[x]/(x^2)) \to \mathbb{A}^1_{\mathbb{C}}$ correspond to ring maps $\mathbb{C}[y] \to \mathbb{C}[x]/(x^2)$, which are given by $y \mapsto ax + b$ with $a, b \in \mathbb{C}$.

Example 12.5 (Affine line). The affine line $\operatorname{Spec} \mathbb{C}[t]$ has a *generic point* corresponding to the zero ideal. This point is dense in the whole space, and its residue field is $\mathbb{C}[t]_{(0)} = \mathbb{C}(t)$.

Example 12.6 (Affine plane). Similarly, in $\operatorname{Spec} \mathbb{C}[x, y]$, we have three types of points: closed points $(x - \alpha, y - \beta)$; points (f) with f irreducible, whose closure is the irreducible algebraic curve defined by $f(x, y) = 0$; and the generic point (0), whose closure is the whole plane.

Example 12.7 (Local schemes). The affine scheme $\operatorname{Spec} \mathbb{C}[t]_{(t)}$ has two points, an open point (0) and a closed point (t).

13 2015-02-18: Points and schemes

Guest lecture by Daniel Erman.

13.1 Examples of affine schemes

We now study points of affine schemes. Let R be a commutative ring. A point $P \in \operatorname{Spec} R$ is maximal if and only if $\{P\}$ is closed, in which case P is a *closed point*. On the other hand, if R is an integral domain, the zero ideal is dense in the whole space and is called the *generic point*. More generally, if P is not closed, then we say P is the generic point of $\overline{\{P\}}$.

Example 13.1 (Real affine line). Consider $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[t]$. There are three types of points:

- Closed points with residue field $\mathbb{R}$, corresponding to ideals $(t - a)$ for $a \in \mathbb{R}$.
- Closed points with residue field $\mathbb{C}$, corresponding to ideals $(t^2 + bt + c)$ with $b^2 - 4c < 0$. These can be thought of as conjugate pairs of points of $\mathbb{A}^1_{\mathbb{C}}$.
- The generic point of $\mathbb{A}^1_{\mathbb{R}}$, corresponding to (0).

Example 13.2 (Complex affine plane). Consider the affine plane $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$. There are three types of points:

- By the Nullstellensatz, the closed points correspond exactly to ideals $(x - \alpha, y - \beta)$ with $\alpha, \beta \in \mathbb{C}$.
- Generic points of irreducible curves, corresponding to (f) with f irreducible.
- The generic point of $\mathbb{A}^2_{\mathbb{C}}$.

Example 13.3 (Local rings). Consider the map $\operatorname{Spec} \mathbb{C}[t]_{(t)} \to \operatorname{Spec} \mathbb{C}[t]$. This is the "Zariski-local" picture of the affine line at a point.

Remark 13.4. In general, given a ring map $\phi : A \to B$, the corresponding scheme map $\operatorname{Spec} B \to \operatorname{Spec} A$ is given by $P \mapsto \phi^{-1}(P)$.

13.2 Locally ringed spaces

We want to consider $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ not just as a ringed space, but with the structure of a *locally* ringed space.

Definition 13.5. A *locally ringed space* is a ringed space whose stalks are all local rings. A morphism of locally ringed spaces is a morphism of ringed spaces such that the stalks of the structure morphism preserves the maximal ideals.

The key idea is that specifying a sheaf on a basis for the topology uniquely defines the sheaf. An affine scheme $\operatorname{Spec} R$ has a basis of basic open affines $(\operatorname{Spec} R_f)_{f\in R}$. Indeed, for any ideal $I \subseteq R$, $\operatorname{Spec} R \setminus V(I) = \bigcup_{f\in I} \operatorname{Spec} R_f$. One can verify that $\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R_f) = R_f$ and that this satisfies the sheaf axioms. Hence, for $Pin \operatorname{Spec} R$,

$$\mathcal{O}_{X,P} = \operatorname*{colim}_{U \ni P} \mathcal{O}_X(U) = \operatorname*{colim}_{\operatorname{Spec} R_f \in P} \mathcal{O}_X(R_f) = \operatorname*{colim}_{f \notin P} R_f = R_P.$$

This is a local ring, so $\operatorname{Spec} R$ is a locally ringed space.

Definition 13.6. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which has an open cover by affine schemes.

Remark 13.7. If $U \subseteq X$ is an open subset, then $\mathcal{O}_U = \mathcal{O}_X|_U$ is also a locally ringed space. So open immersions of schemes correspond to open subsets.

Remark 13.8. Let $U \subseteq \operatorname{Spec} A$ be an open subset. Then U is covered by basic affine open subsets, and hence is a scheme (not necessarily affine).

Example 13.9 (A punctured plane). The scheme $\mathbb{A}^2_{\mathbb{C}} \setminus V(x,y)$ has an open affine cover by $\operatorname{Spec} \mathbb{C}[x, x^{-1}, y]$ and $\operatorname{Spec} \mathbb{C}[x, y, y^{-1}]$, but is not affine.

14 2015-02-20: Gluing and morphisms

Guest lecture by Daniel Erman.

14.1 Gluing

Proposition 14.1 (Gluing schemes). *Let $\{X_i\}_{i\in I}$ be a family of schemes. For each $i, j \in I$, let X_{ij} be an open subscheme of X_i. Let $\varphi_{ij} : X_{ij} \xrightarrow{\simeq} X_{ji}$ be isomorphisms such that, for all $i, j, k \in I$:*

(1) $\varphi_{ij} \circ \varphi_{ji} = \mathrm{id}_{X_{ij}}$.

(2) $\varphi_{ik}(X_{ij} \cap X_{ik}) = X_{ki} \cap X_{kj}$.

(3) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij} : X_{ij} \cap X_{ik} \xrightarrow{\simeq} X_{ki} \cap X_{kj}$.

Then there exists a unique scheme X with an open cover $\{U_i\}_{i\in I}$ of X and isomorphisms $\psi_i : X_i \xrightarrow{\simeq} U_i$ such that $\psi_i|_{X_{ij}} = \psi_j|_{X_{ji}} \circ \varphi_{ij} : X_{ij} \to U_i \cap U_j$ for all $i, j \in I$.

Example 14.2. Let A and B be commutative rings. Let $\operatorname{Spec} A' \subset \operatorname{Spec} A$ and $\operatorname{Spec} B' \subset \operatorname{Spec} B$ be open subsets. Let $f : A' \xrightarrow{\simeq} B'$ be a ring isomorphism, inducing an isomorphism $\phi : \operatorname{Spec} B' \to \operatorname{Spec} A'$. Then we can glue $\operatorname{Spec} A$ and $\operatorname{Spec} B$ along ϕ.

Example 14.3. Gluing $\operatorname{Spec} k[t]$ and $\operatorname{Spec} k[s]$ along $t \mapsto s : \operatorname{Spec} k[t, t^{-1}] \xrightarrow{\simeq} \operatorname{Spec} k[s, s^{-1}]$ yields an "affine line with two origins", a non-separated line X. A global section of X is a polynomial $f(t)$ on one patch and $f(s)$ on the other.

Example 14.4. Gluing $\operatorname{Spec} k[t]$ and $\operatorname{Spec} k[s]$ along $t \mapsto s^{-1} : \operatorname{Spec} k[t, t^{-1}] \xrightarrow{\simeq} \operatorname{Spec} k[s, s^{-1}]$ yields the projective line $\mathbb{P}^1_k$, the global sections of which are all constant.

Remark 14.5. One can also glue along closed subschemes. This allows us to construct a scheme with no closed points by gluing countably many copies of a DVR to itself.

14.2 Morphisms

Definition 14.6. A *morphism of schemes* $X \to Y$ is a morphism of locally ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

Remark 14.7. We can glue compatible families of morphisms of affine schemes. In fancier language, affine morphisms satisfy descent for the Zariski topology (and, in fact, for the fpqc topology).

15 2015-02-23: The functor of points

Definition 15.1. Let R be a commutative ring. By definition, an R-point of a scheme S is a morphism $\operatorname{Spec} R \to S$. This defines a functor $R \mapsto \operatorname{Hom}_{\mathbf{Sch}}(\operatorname{Spec} R, S) : \mathbf{Ring} \to \mathbf{Set}$, called the *functor of points* of S.

Example 15.2. Let k be a field, and write $\operatorname{Spec} k = \{*\}$. A k-point of a ringed space $(S, \mathcal{O}_S)$ is a morphism of ringed spaces $(f, f^\sharp) : (\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k}) \to (S, \mathcal{O}_S)$, which is equivalent to giving a point $x = f(*)$ and a ring map $f^\sharp_x : \mathcal{O}_{S,x} \to k$.

If $(S, \mathcal{O}_S)$ is a locally ringed space, then $\mathcal{O}_{S,x}$ is a local ring, and $f^\sharp_x$ must map the maximal ideal $\mathfrak{m}_{S,x}$ into $(0) \subset k$, i.e., $f^\sharp_x$ induces an embedding of fields $\kappa_x \hookrightarrow k$, where $\kappa_x = \mathcal{O}_{S,x}/\mathfrak{m}_{S,x}$ is the residue field of S at x. In particular, a k-point of a scheme S is a point $x \in S$ together with an embedding $\kappa_x \hookrightarrow k$.

Remark 15.3. Each point $x \in S$ can naturally be viewed as a κ_x-point. For this reason, κ_x is sometimes called the *field of definition* of x. The field of definition has the universal property that for any field k, any morphism $\operatorname{Spec} k \to S$ with image $\{x\}$ uniquely factors through the natural morphism $\operatorname{Spec} \kappa_x \to S$.

Remark 15.4. Let $U \subset S$ be an open subset. A section $f \in \mathcal{O}_S(U)$ can be viewed as a "function" on S whose value at $x \in U$ is the image of f in $\kappa_x = \mathcal{O}_{S,x}/\mathfrak{m}_{S,x}$. Equivalently, this is the pullback under $\operatorname{Spec} \kappa_x \to S$.

Example 15.5. Suppose $S = \operatorname{Spec} R$ is an affine scheme. Let $\mathfrak{p}_x \subset R$ be the prime ideal corresponding to a point $x \in S$. For $f \in R$, we have $f(x) = 0 \in \kappa_x \iff f \in \mathfrak{p}_x$. So we can once again interpret closed subsets of S as being cut out by systems of equations:

$$V(T) = \{x \in S : f(x) = 0 \ \forall x \in T\}.$$

Note 15.6. If $f(x) = 0$ for all $x \in \operatorname{Spec} R$, then $f \in \bigcap_{\mathfrak{p}} \mathfrak{p}$ is nilpotent. So nilpotent elements aren't visible in this function perspective.

16 2015-02-25: The topology of schemes

16.1 k-points and systems of equations

Definition 16.1. A *geometric point* of a scheme S is a k-point, where k is an algebraically closed field.

Example 16.2. The scheme $\operatorname{Spec} \mathbb{Z}[t]/(t^2+1)$ has two $\mathbb{C}$-points, given by $t \mapsto \pm i$.

Example 16.3. More generally, R-points of $\operatorname{Spec} \mathbb{Z}[t_1, \ldots, t_n, \ldots]/(f_1, \ldots, f_m, \ldots)$ correspond to solutions in R of the system of equations $0 = f_1 = \cdots = f_m = \ldots$ in $t_1, \ldots, t_n, \ldots$.

Remark 16.4. $\operatorname{Spec} \mathbb{Z}$ is the terminal object in the category of schemes.

16.2 Closed subsets

Let $S = \operatorname{Spec} R$ be an affine scheme. To each ideal $I \subset R$, we associate a closed subset $Z(I) = \{x : \mathfrak{p}_x \supset I\} = Z(I)$; and to each subset $X \subset S$, we associate an ideal $I(X) = \{f \in R : f|_X = 0\} = \bigcap_{x \in X} \mathfrak{p}_x$. This correspondence has the following properties:

(1) $Z(I(X)) = \overline{X}$.

(2) $I(Z(J)) = \sqrt{J} = \bigcap_{\mathfrak{p} \supset J} \mathfrak{p}$.

(3) $Z(\sum_\alpha I_\alpha) = \bigcap_\alpha Z(I_\alpha)$.

(4) $Z(J_1 J_2) = Z(J_1 \cap J_2) = Z(J_1) \cup J(Z_2)$.

Hence, this restricts to a bijection between radical ideals of R and closed subsets of S.

Remark 16.5. Let X_{var} be an affine variety over an algebraically closed field k. Let $R = k[X_{\text{var}}]$ be its coordinate ring. Consider the affine scheme $X_{\text{sch}} = \operatorname{Spec} R$. As topological spaces, we have an inclusion $X_{\text{var}} \subset X_{\text{sch}}$. Points of X_{sch} correspond to irreducible closed subsets of X_{var}. (As topological spaces, X_{sch} is the *soberification* of X_{var} — every continuous map from X_{var} into a sober space uniquely factors through the inclusion $X_{\text{var}} \subset X_{\text{sch}}$.)

Remark 16.6. Let $S = \operatorname{Spec} R$ be an affine scheme. Let $x \in S$ be a point corresponding to a prime ideal $\mathfrak{p}_x \subset R$. Then $\overline{\{x\}} = Z(\mathfrak{p}_x) = \{y : \mathfrak{p}_y \supset \mathfrak{p}_x\}$. In particular, $\overline{\{x\}} = \{x\}$ iff $\mathfrak{p}_x$ is maximal, so we can recover the maximal spectrum as the set of closed points in S.

Remark 16.7. The Nullstellensatz implies that, if R is a finitely-generated algebra over a field k, then closed points (which are defined over $\overline{k}$) are dense in $\operatorname{Spec} R$.

Definition 16.8. Let S be a scheme, and let $x, y \in S$. If $x \in \overline{\{y\}}$, then we say x is a *specialization* of y, and y is a *generalization* of x.

A *generic point* of S is a point $x \in S$ such that $\overline{\{x\}} = S$. A scheme S has a generic point if and only if S is irreducible, in which case the generic point is unique.[1]

Remark 16.9. Any affine scheme is quasi-compact.

[1] Contrast with Weil's earlier approach to algebraic geometry, where there are often many generic points.

17 2015-02-27: Topological properties of schemes

Proposition 17.1. *Let $S = \operatorname{Spec} R$ be an affine scheme.*

(1) S is quasi-compact.

(2) S is irreducible iff R has a unique minimal prime iff the nilradical of R is prime iff all zero-divisors of R are nilpotent.

(3) S is connected iff R has no nontrivial idempotents.

(4) If R is a Noetherian ring, then S is a Noetherian space (i.e., satisfies the descending chain condition for closed subsets).[2]

(5) The dimension of S as a topological space (i.e., the supremum of lengths of chains of irreducible closed subsets) is equal to the Krull dimension of R.

Definition 17.2. A topological space S is called *quasi-separated* iff the intersection of two quasi-compact open subsets is quasi-compact.

Remark 17.3. If S is a quasi-separated scheme, and $S = \bigcup_\alpha U_\alpha$ is an affine open cover of S, then $U_\alpha \cap U_\beta$ is a finite union of affine charts for any α, β.

Exercise 17.4. The converse of the above statement is also true.

Example 17.5. The scheme $\operatorname{Spec} k[x_1, x_2, x_3, \dots]$ is non-Noetherian. If we remove the point corresponding to the maximal ideal $(x_1, x_2, x_3, \dots)$, the resulting scheme, which is isomorphic to $\bigcup_i \operatorname{Spec} k[x_1, x_2, \dots][x_i^{-1}]$, is not quasi-compact.

Proposition 17.6. *A scheme S is quasi-compact and quasi-separated if and only if there is a finite affine open cover $S = \bigcup_{i=1}^n U_i$ such that $U_i \cap U_j$ is a finite union of affine open subsets for each i, j.*

18 2015-03-02: Zariski-local properties of schemes

Definition 18.1. Let P be a property of commutative rings. We say P is a *Zariski-local property* provided that:

(1) For all $f \in R$, if R has property P, then $R[f^{-1}]$ has property P.

(2) If $(f_1, \dots, f_k) = (1)$ and each $R[f_i^{-1}]$ has property P, then R has property P.

We say a scheme S has property P provided that every open affine subset of S has property P.

Proposition 18.2. *Let P be a Zariski-local property. Let $S = \bigcup_\alpha U_\alpha$ be a scheme with an affine open cover. Then S has property P if and only if every U_α has property P.*

[2]The converse isn't true in general, since closed subsets of S only correspond to *radical* ideals of R.

It suffices to show that, given affine opens $U_\alpha, V \subset S$, any $x \in U_\alpha \cap V$ has an open neighborhood W that is principal open in U_α and in V. This follows from the following property of principal open subsets:

Lemma 18.3. *Suppose $S = \operatorname{Spec} R$ is affine and $U \subset S$ is a principal affine open. Then an open subset $W \subset U$ is principal as an open subset of U if and only if W is principal as an open subset of S.*

Moreover, if U is any affine open subset (not necessarily principal), then a subset of U that is principal open in S is principal open in U.

Proof. If $S \supset U \supset W$ and $W = S - Z(f)$, then $W = U - Z(f)$. If $S \supset U \supset W$ and $U = S - Z(f)$ and $W = U - Z(g)$, then $W = S - Z(fg)$. $\square$

Example 18.4. Here are some Zariski-local properties of commutative rings: reduced, integral, regular, normal, Jacobson, Noetherian.

Some Zariski-local properties follow a different pattern of terminology when extended to schemes. For such a property P, we say a scheme S has the property "locally P" if S has an affine open covering by affine schemes with property P, and we say S has property P if S is locally P *and* quasi-compact. For example:

Definition 18.5. A *locally Noetherian* scheme S is a scheme such that every affine open subset of S is the spectrum of a Noetherian ring. A *Noetherian* scheme is a quasi-compact, locally Noetherian scheme.

19 2015-03-04: Embeddings

19.1 More properties of schemes

Definition 19.1. A locally Noetherian scheme S is *regular* iff all stalks of S are regular local rings.

Exercise 19.2. A scheme S is reduced iff all stalks of $\mathcal{O}_S$ are reduced.

Remark 19.3. A connected scheme whose stalks are all integral domains is *not necessarily* integral: there is a connected reducible scheme S whose stalks are all integral domains. See [Stacks, tag 0568].

19.2 Open embeddings

Definition 19.4. Let S be a scheme, and let $U \subset S$ be an open subset. Then $(U, \mathcal{O}_S|_U)$ is an *open subscheme* of S, and the inclusion $j : U \hookrightarrow S$ is called an *open embedding*.

Abstractly, a morphism of schemes $f : X \to Y$ is an open embedding iff f induces an isomorphism $X \xrightarrow{\simeq} f(X)$, and $f(X)$ is an open subscheme of Y.

Example 19.5. The inclusion of the generic point $\operatorname{Spec} k(x) \to \operatorname{Spec} k[x]_{(x)}$ is an open embedding. However, the inclusion $\operatorname{Spec} k(x) \to \operatorname{Spec} k[x] = \mathbb{A}^1_k$ is *not* an open embedding, because the generic point isn't open in $\mathbb{A}^1_k$.

19.3 Closed embeddings of affine schemes

Definition 19.6. Let $S = \operatorname{Spec} R$ be an affine scheme. A *closed subscheme* of S is $\operatorname{Spec}(R/I)$ for an ideal $I \subset R$. The quotient map $R \twoheadrightarrow R/I$ induces a map $i : \operatorname{Spec}(R/I) \hookrightarrow \operatorname{Spec} R$.

Note that the set-theoretic image of i is $Z(I)$, which only depends on the radical $\sqrt{I}$. So a closed subscheme is a closed subset *plus* some extra structure. Moreover, $\operatorname{Spec}(R/\sqrt{I})$ is the unique *reduced* closed subscheme of S corresponding to the same closed subset.

Example 19.7. Let k be an algebraically closed field. Consider the scheme $\operatorname{Spec} k[x, y]/(x^2 - a, y) \subset \mathbb{A}^2$. For $a \neq 0$, this is the disjoint union of two points, and is reduced. But for $a = 0$, this corresponds to the ideal (x^2, y), which is a single non-reduced point.

Example 19.8. Consider the scheme $\operatorname{Spec} k[x, y]/(xy, y^2)$. This is like $\mathbb{A}^1$, but it "remembers a tangent vector" at the origin.[3]

Next time, we will extend this to arbitrary schemes.

20 2015-03-06: Closed embeddings

Definition 20.1. A *closed embedding* is a morphism of schemes $i : X \hookrightarrow S$ such that, for any affine open $U = \operatorname{Spec} R \subset S$, there is an ideal $I \subset R$ such that $i^{-1}(U) = \operatorname{Spec}(R/I)$.

Fact 20.2. It suffices to verify this for some affine open cover of S.

Definition 20.3. Let S be a scheme. Given any closed embedding $i : X \hookrightarrow S$, define the *ideal sheaf* of the closed embedding to be the sheaf $\mathscr{I}_{X/S} \subset \mathcal{O}_S$ given on each open $U \subset S$ by

$$\mathscr{I}_{X/S}(U) = \left\{ f \in \mathcal{O}_S(U) : i^{-1}(f) = 0 \right\}.$$

Note that $\mathscr{I}_{X/S}(U)$ is an ideal in $\mathcal{O}_S(U)$. We will give an intrinsic definition of ideal sheaves later.

Example 20.4. If $S = \operatorname{Spec} R$ is an affine scheme and $X = \operatorname{Spec}(R/I)$, then $\mathscr{I}_{X/S}(S) = I$. More generally, if $g \in R$ and $U = D(g)$ is the corresponding principal open subset, then $\mathscr{I}_{X/S}(U) = \ker(R[g^{-1}] \to (R/I)[g^{-1}]) = I[g^{-1}]$.

Lemma 20.5. *The ideal sheaf $\mathscr{I} \subset \mathcal{O}_S$ of a closed embedding $i : X \hookrightarrow S$ satisfies the following condition:*

(*) *For any affine open $U = \operatorname{Spec} R \subset S$ and any principal open $V = \operatorname{Spec} R[g^{-1}] \subset U$, we have $\mathscr{I}(V) = \mathscr{I}(U)[g^{-1}]$.*

Conversely, given a sheaf $\mathscr{I} \subset \mathcal{O}_S$ locally given by ideals and satisfying the above criterion, we can construct a subscheme $X \subset S$ corresponding to $\mathscr{I}$ as follows: let $S = \bigcup_\alpha U_\alpha$ be an open cover by affine open subschemes $U_\alpha = \operatorname{Spec} R_\alpha$, and let $I_\alpha = \mathscr{I}(U_\alpha) \subset R_\alpha$. Then $X = \bigcup_\alpha \operatorname{Spec}(R_\alpha/I_\alpha)$.

Not all sheaves locally given by ideals come from closed subschemes. Here is one such sheaf:

[3] This is an example of an "embedded point"; see [EH] for more discussion.

Example 20.6. Let $S = \mathbb{A}^1 = \operatorname{Spec} k[x]$ and $U = \mathbb{A}^1 \setminus \{0\} = \operatorname{Spec} k[x, x^{-1}]$. Let $j : U \hookrightarrow \mathbb{A}^1$ be the open embedding. Then $j_!(\mathcal{O}_U) \subset \mathcal{O}_{\mathbb{A}^1}$ is the sheaf[4] given by

$$j_!(\mathcal{O}_U)(V) = \begin{cases} \mathcal{O}_{\mathbb{A}^1}(V) & \text{if } V \subset U, \\ 0 & \text{if } 0 \in V. \end{cases}$$

Informally, one can write $j_!(\mathcal{O}_U) = \{f \in \mathcal{O}_{\mathbb{A}^1} : f_0 = 0 \in \mathcal{O}_{\mathbb{A}^1,0}\}$. Even though $j_!(\mathcal{O}_U)$ is a sheaf of $\mathbb{A}^1$ locally given by ideals, it does not correspond to a closed subscheme of $\mathbb{A}^1$.

21 2015-03-09: Ideal sheaves

21.1 Ideal sheaves and closed embeddings

Definition 21.1. Let S be a scheme. An *ideal sheaf* $\mathscr{I} \subset \mathcal{O}_S$ is a sheaf of ideals satisfying

(**) For any affine open $U = \operatorname{Spec} R \subset S$ and any principal open $D(g) = \operatorname{Spec} R[g^{-1}] \subset U$, the natural map $\mathscr{I}(U)[g^{-1}] \to \mathscr{I}(D(g))$ is an isomorphism.

Exercise 21.2. It suffices to check (*) and (**) for some cover by affine open schemes.

Proposition 21.3. *Let $\mathscr{I}$ be an ideal sheaf on a scheme S. Then there exists a closed embedding $i : X \to S$ such that $\mathscr{I} = \ker(\mathcal{O}_S \to i_*\mathcal{O}_X)$. This closed embedding is unique up to natural isomorphism, giving a natural equivalence between ideal sheaves and closed embeddings.*

Example 21.4. Let $S = \operatorname{Spec} k[x]$ and $\mathscr{I} = x \cdot \mathcal{O}_S \subset \mathcal{O}_S$. If $0 \notin U$, then $\mathscr{I}(U) = \mathcal{O}_S(U)$. If $0 \in U$, then $\mathscr{I}(U) = x \cdot \mathcal{O}_S(U) \subsetneq \mathcal{O}_S(U)$. So $\mathcal{O}_S/\mathscr{I}$ is the skyscraper sheaf with stalk k at 0.

Definition 21.5. Let F be a sheaf on a space S. The *support* of F, denoted $\operatorname{supp}(F)$, is the smallest closed subset $Z \subset S$ such that $F|_{S \setminus Z} = 0$.

In general, the closed embedding $i : X \to S$ corresponding to an ideal sheaf $\mathscr{I} \subset \mathcal{O}_S$ can be explicitly constructed as follows: Put $X = \operatorname{supp}(\mathcal{O}_S/\mathscr{I})$, let $i : X \hookrightarrow S$ be the inclusion, and put $\mathcal{O}_X = i^{-1}(\mathcal{O}_S/\mathscr{I})$, so that $\mathcal{O}_S/\mathscr{I} = i_*\mathcal{O}_X$. There are lots of things to check, which we omit. $\square$

Exercise 21.6. If S is affine and $i : X \hookrightarrow S$ is a closed embedding, then X is affine.

Problems like this will become much easier once we've developed a systematic treatment of quasi-coherent sheaves.

Definition 21.7. A *locally closed subscheme* (or simply a *subscheme*) of S is a closed subscheme of an open subscheme of S. A *locally closed embedding* is a composition of a closed embedding and an open embedding.

[4] In the classical topology, we have to sheafify. However, in the Zariski topology, this already gives a sheaf.

21.2 Morphisms as families

We can think of a morphism of schemes $f : X \to S$ as a family of schemes parametrized by S, the elements of the family being the fibers of f.

Example 21.8. A k-algebra is a ring R together with a ring map $k \to R$. A k-ringed space is a ringed space X together with a morphism $X \to \operatorname{Spec} k$.

Example 21.9. Let $X = \operatorname{Spec} R$ be an affine scheme. Since $\operatorname{Spec} \mathbb{Z}$ is the terminal object in the category of schemes, there is a unique morphism $X \to \operatorname{Spec} \mathbb{Z}$. The fibers of this morphism are $\operatorname{Spec}(R \otimes_{\mathbb{Z}} \mathbb{F}_p) = \operatorname{Spec}(R/pR)$ over the closed points $(p) \in \operatorname{Spec} \mathbb{Z}$, and $\operatorname{Spec}(R \otimes_{\mathbb{Z}} \mathbb{Q})$ over the generic point (0).

22 2015-03-11: Local properties of morphisms

The general philosophy is to study *relative* properties (properties of morphisms), not just *absolute* properties (properties of schemes).

Definition 22.1 (Properties of morphisms local over the base)**.** Let P be one of the following properties: affine, quasi-compact, quasi-separated, semi-separated[5], separated. We say a morphism $f : X \to Y$ has property P if and only if for any affine open $U \subset Y$, $f^{-1}(U)$ has property P. See [Stacks, tag 01QL] for more properties of morphisms.

Claim 22.2. *It suffices to check this for some affine open cover of Y.*

A different pattern is for a property of morphisms to be *local over the source*. For example:

Definition 22.3. A morphism $f : X \to Y$ is *locally of finite type* if for any affine opens $U = \operatorname{Spec} A \subset X$ and $V = \operatorname{Spec} B \subset Y$ such that $f(U) \subset V$, A is finitely-generated as a B-algebra.

Claim 22.4. *It suffices to check this for some affine open covers $\{U_\alpha \subset X\}$ and $\{V_\alpha \subset Y\}$ such that $f(U_\alpha) \subseteq V_\alpha$ and $\bigcup_\alpha U_\alpha = X$.*

Definition 22.5. A morphism f is of *finite type* if and only if f is locally of finite type and quasi-compact.

Remark 22.6. A morphism $f : X \to Y$ being of finite type means that for any affine open $V \subset Y$, $f^{-1}(V) = \bigcup_{i=1}^n \operatorname{Spec} A_i$ for some finitely-generated B-algebras A_i.

We may intuitively think of finite-type morphisms as being given locally (on the base) by a "finite amount of data". This is true when the base is locally Noetherian, but for arbitrary schemes, a better-behaved notion is (locally) finite *presentation*, where the defining ideals are also required to be finitely-generated.

Definition 22.7 (Relative schemes)**.** Let Y be a scheme. A *scheme over* Y (or an Y*-scheme*) is a morphism of schemes $f : X \to Y$. We also sometimes write X/Y. (If A is a commutative ring, "scheme over $\operatorname{Spec} A$" may be abbreviated to "scheme over A".) In other words, the category of Y-schemes is the slice category $\mathbf{Sch}/Y$.

[5]A scheme is called *semi-separated* if the intersection of any two affine open subsets is affine.

We will often refer to properties of morphisms of schemes as properties of relative schemes. For example, we will say "X is affine over Y" to mean that $f : X \to Y$ is an affine morphism. (The phrase "X is an affine Y-scheme" is synonymous, but has more potential for confusion, so we will avoid it.)

22.1 Varieties

Definition 22.8. Let k be a field. A *variety* over k is a geometrically[6] reduced scheme locally of finite type over k. More explicitly, a variety is a morphism of schemes $f : X \to \operatorname{Spec} k$ which is locally of finite type, hence X has an open cover $X = \bigcup_\alpha U_\alpha$, where each $U_\alpha = \operatorname{Spec} A_\alpha$ is an affine scheme with A_α a reduced, finitely-generated k-algebra, and the gluing maps are k-linear.

A morphism of varieties is just a morphism in $\mathbf{Sch}/k$ between varieties. In other words, the category of k-varieties $\mathbf{Var}/k$ embeds fully faithfully into $\mathbf{Sch}/k$.

What does this look like for varieties over a non-algebraically closed field, such as $\mathbb{R}$? Classically, we think of an affine variety over $\mathbb{R}$ as a subset of $\mathbb{C}^n$ cut out by polynomial equations with real coefficients. Maps of real varieties (including gluing maps) must be polynomials with real coefficients.

Schematically, given an $\mathbb{R}$-variety $X \to \operatorname{Spec} \mathbb{R}$, we can look at the real points ($\mathbb{R}$-morphisms $\operatorname{Spec} \mathbb{R} \to X$) or the complex points ($\mathbb{R}$-morphisms $\operatorname{Spec} \mathbb{C} \to X$).

Example 22.9. The affine line $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x]$, considered as an $\mathbb{R}$-scheme, has $\mathbb{R}$-points corresponding to $(x - c)$ for $c \in \mathbb{R}$, and has $\mathbb{C}$-points corresponding to irreducible quadratic polynomials in $\mathbb{R}[x]$.

23 2015-03-13 through 2015-03-18

[I missed these lectures due to the Arizona Winter School.]

24 2015-03-20: Separated morphisms

Definition 24.1. A morphism $f : X \to Y$ of schemes is called *separated* if the diagonal $\Delta_f : X \to X \times_Y X$ is a closed embedding.

Example 24.2. Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes. Then $\Delta_f : \operatorname{Spec} A \to \operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} A = \operatorname{Spec}(A \otimes_B A)$ corresponds to the multiplication map $A \otimes_B A \to A$, which is surjective. Hence Δ_f is a closed embedding, so f is separated.

Example 24.3. Let X be the affine line over k with two origins, i.e., $X = U_1 \cap U_2$ with $U_1 = U_2 = \operatorname{Spec} k[t]$, and $U_1 \cap U_2$ glued along the identity map. Then $X \times_k X = \bigcup_{i,j=1,2} U_i \times_k U_j$. The preimage in $X \times X$ of the diagonal in $\mathbb{A}^2$ is a diagonal line in each $U_i \times U_j$, and hence is a k-line with *four* origins, corresponding to elements of $\{1, 2\}^2$. However, the image of

[6] If k is not a perfect field, then even if $k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ is reduced, $\overline{k}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ might not be reduced. We will discuss this in more detail later. Note that over a perfect field, there is no difference between *geometrically reduced* and *reduced*.

$\Delta : X \to X \times_k X$ only contains *two* of these origins, hence is non-closed. Hence, the structure map $X \to \operatorname{Spec} k$ is not separated.

Remark 24.4. This has a topological analogue: if X is a topological space, then X is Hausdorff if and only if the diagonal map $\Delta : X \hookrightarrow X \times X$ is a closed embedding.

Proposition 24.5. *(1) Separatedness is local over the target.*

(2) Separatedness is stable under base change.

Proof. (1) A cover of Y gives compatible covers of X and $X \times_Y X$. One can locally check the property of being a closed embedding.

(2) Everything base-changes to $Z \to Y$ in a compatible way. □

Corollary 24.6. *Any affine morphism is separated.*

Remark 24.7. Any locally closed embedding is affine. Moreover, for any scheme X, the natural map $\mathbb{A}^n_X = X \times \mathbb{A}^n \to X$ is affine. In practice, many affine morphisms arise as a composition $Y \hookrightarrow \mathbb{A}^n_X \to X$, where Y is a closed subscheme of $\mathbb{A}^n_X$.

Exercise 24.8. Compositions of separated maps are separated.

Let us give a more explicit description of separatedness. Since separatedness is local over the base, consider a morphism $f : X \to \operatorname{Spec} B$ and write $X = \bigcup_\alpha U_\alpha$, where $U_\alpha = \operatorname{Spec} A_\alpha$. Then $X \times_Y X = \bigcup_{\alpha,\beta} U_\alpha \times_B U_\beta = \bigcup_{\alpha,\beta} \operatorname{Spec}(A_\alpha \otimes_B A_\beta)$. Consider

$$\Delta_f : X \to X \times_B X = \bigcup_{\alpha,\beta} U_\alpha \times_B U_\beta.$$

Then $\Delta_f^{-1}(U_\alpha \times_B U_\beta) = U_\alpha \cap U_\beta$. The map f is separated if and only if $U_\alpha \cap U_\beta \to U_\alpha \times_Y U_\beta$ is a closed embedding for all α, β. (In particular, $U_\alpha \cap U_\beta$ must be affine, so f is semi-separated.)

Remark 24.9. To summarize separatedness properties: a morphism $f : X \to Y$ is

- *separated* if and only if $\Delta_f : X \to X \times_Y X$ is a closed embedding.
- *semi-separated* if and only if $\Delta_f : X \to X \times_Y X$ is affine.
- *quasi-separated* if and only if $\Delta_f : X \to X \times_Y X$ is quasi-compact.

25 2015-03-23: Separated morphisms, continued

Proposition 25.1. *Let $f : X \to Y$ be a morphism of schemes. The corresponding diagonal map $\Delta_f : X \to X \times_Y X$ is a locally closed embedding.*

In other words, the question of whether a morphism is separated is purely topological: f is separated if and only if $\Delta_f(X) \subset X \times_Y X$ is closed in the Zariski topology.

Remark 25.2. Assume $Y = \operatorname{Spec} B$ is affine. Given an open cover $X = \bigcup_\alpha U_\alpha$ by affine schemes $U_\alpha = \operatorname{Spec} A_\alpha$, Δ_f is closed if and only if $U_\alpha \cap U_\beta \hookrightarrow U_\alpha \times_Y U_\beta$ is closed for all α, β. Consider the composition

$$U_\alpha \cap U_\beta \hookrightarrow U_\alpha \times_Y U_\beta \hookrightarrow U_\alpha \times_{\operatorname{Spec} \mathbb{Z}} U_\beta.$$

Note that $U_\alpha \times_Y U_\beta$ is a closed subscheme of $U_\alpha \times_{\operatorname{Spec}\mathbb{Z}} U_\beta$; this is because $A_\alpha \otimes_{\mathbb{Z}} A_\beta \twoheadrightarrow A_\alpha \otimes_B A_\beta$ is surjective. Hence, $U_\alpha \cap U_\beta$ is closed in $U_\alpha \times_Y U_\beta$ if and only if $U_\alpha \cap U_\beta$ is closed in $U_\alpha \times_{\operatorname{Spec}\mathbb{Z}} U_\beta$.

Corollary 25.3. *Suppose Y is affine. Then $X \times_Y X \hookrightarrow X \times_{\operatorname{Spec}\mathbb{Z}} X$ is a closed embedding. Therefore, $\Delta_f : X \hookrightarrow X \times_Y X$ is a closed embedding if and only if $\Delta : X \hookrightarrow X \times_{\operatorname{Spec}\mathbb{Z}} X$ is a closed embedding. In other words, f is separated if and only if X is separated.*

Corollary 25.4. *A morphism of schemes $f : X \to Y$ is separated if and only if for any affine open $V \subset Y$, the preimage $f^{-1}(V)$ is a separated scheme. Moreover, it suffices to check this for an affine cover.*

Here's another approach to separatedness. A scheme X is separated if and only if, for any scheme S and any morphism $S \to X \times X$, the base change $f_S : X \times_{X \times X} S \to S$ is a closed embedding.

Given a "test scheme" S, a map $f : S \to X \times X$ is a pair of morphisms $f_1, f_2 : S \to X$. What is the scheme $X \times_{\Delta,(X\times X),f} S = X \times_{X \times X} S$? As a set, $X \times_{X\times X} S = \{s \in S : f_1(s) = f_2(s)\}$.

The scheme X is separated if and only if for any S and any maps $f_1, f_2 : S \to X$, the subset $\{s \in S : f_1(s) = f_2(s)\} \subset S$ is closed.

Example 25.5. Let $X = \mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t]$. Let S be a scheme, and choose $f_1, f_2 \in \Gamma(S, \mathcal{O}_S)$. Then $\{s \in S : f_1(s) = f_2(s)\} = Z(f_1 - f_2)$ is closed, so $\mathbb{A}^1_{\mathbb{Z}}$ is separated.

Let us reformulate separatedness once more.

Proposition 25.6. *A scheme X is separated if and only if each map $f : U \to X$ defined on a dense locally closed subscheme $U \subset S$ has at most one extension to S.*

26 2015-03-25: Proper morphisms

Definition 26.1. Let $f : X \to Y$ be a morphism of schemes. We say f is *universally closed* if for any morphism $g : Z \to Y$, the base-changed morphism $f_Z : X \times_Y Z \to Z$ is closed. A morphism $f : X \to Y$ is *proper* if f is of finite type, separated, and universally closed.

Example 26.2. Let $g : X \to Z$ be a morphism. Consider its graph Γ_g, which is the preimage of Δ_Z under the morphism $(g, \mathrm{id}_Z) : X \times Z \to Z \times Z$. If Z is separated, then Γ_g is closed. The projection of Γ_g onto Z is the image $g(X)$. So, if X is proper and Z is separated, then for any map $g : X \to Z$, the image $g(X)$ is closed.

Example 26.3. The map $\mathbb{A}^1 \to \mathbb{A}^0$ is closed, but not universally closed: it base-changes to the projection $\mathbb{A}^2 \to \mathbb{A}^1$, and the image of $\{xy = 1\} \subset \mathbb{A}^2$ under this map is $\{x \neq 0\} \subset \mathbb{A}^1$, which is not closed.

Proposition 26.4 (Formal properties of proper morphisms)**.**

(1) Properness is Zariski-local on the base.

(2) Properness is stable under base change.

(3) Any composition of proper morphisms is proper.

Example 26.5. Closed embeddings are proper.

Example 26.6 (Projective space). $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is proper. What is $\mathbb{P}^n_{\mathbb{Z}}$? Consider charts $U_0, \ldots, U_n$, where $U_i = \operatorname{Spec} \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$, glued by identifying $\operatorname{Spec} \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{x_i}{x_j}] \subset \operatorname{Spec} U_i$ with $\operatorname{Spec} \mathbb{Z}[\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}, \frac{x_j}{x_i}] \subset \operatorname{Spec} U_j$ in the obvious way.

27 2015-03-27: Projective morphisms

Theorem 27.1. $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ *is proper.*

Definition 27.2. A morphism $f : X \to Y$ is *projective* if f can be written as a composition $X \hookrightarrow Y \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}} \to Y$, where $X \hookrightarrow Y$ is a closed embedding.

Corollary 27.3. *Projective morphisms are proper.*

Remark 27.4. There is also a weaker notion of projectivity: existence of an open cover $Y = \bigcup_\alpha U_\alpha$ such that each $U_\alpha \times_Y X \to U_\alpha$ is projective.

Definition 27.5. A morphism $f : X \to Y$ is *quasi-projective* if f can be written as a composition $X \hookrightarrow Y \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}} \to Y$, where $X \hookrightarrow Y$ is a locally closed embedding.

Corollary 27.6. *Quasi-projective morphisms are separated.*

Example 27.7. Let $X = Z(f_1, \ldots, f_n) \subset \mathbb{P}^n$. Write $D_i = \deg(f_i)$ and

$$f_i(x_0, \ldots, x_n) = \sum_{\sum_j d_j = D_i} a_{d_0 \ldots d_n} x_0^{d_0} \cdots x_n^{d_n}.$$

Consider the coefficients $a_{d_0 \ldots d_n}$ as coordinates in $\mathbb{A}^M$ (where M is the total number of coefficients in all f_i). We can define the "universal scheme" $\mathscr{X} \subset \mathbb{P}^n \times \mathbb{A}^M$ as $Z(f_1, \ldots, f_m)$ for "universal" f_i (i.e., with indeterminate coefficients). The projection map $\pi : \mathscr{X} \to \mathbb{A}^M$ is projective.

Given a property P of schemes, we can consider

$$\left\{ \bar{a} \in \mathbb{A}^M : \pi^{-1}(\bar{a}) = X_{\bar{a}} \text{ has property } P \right\}.$$

If this is closed in the Zariski topology on $\mathscr{X}$, we say P is a *closed property*. Many commonly studied properties of schemes are closed.

28 2015-04-06: Projective space is proper

Recall that a *proper morphism* is a morphism of schemes which is of finite type, separated, and universally closed. Today we prove that projective space is proper.

Theorem 28.1. *$\mathbb{P}^N$ is proper. (That is, the unique morphism $\mathbb{P}^N_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is proper.)*

It is clear that $\mathbb{P}^N$ is of finite type. For separatedness, we need to show that the diagonal $\Delta \subset \mathbb{P}^N \times \mathbb{P}^N$ is closed. This can be checked in charts, or we can write Δ as the zero locus of bihomogeneous polynomials $x_i y_j - x_j y_i$, where $i, j \in \{0, \ldots, N\}$.

More precisely, given a k-valued point $\operatorname{Spec} k \to \mathbb{P}^N \times \mathbb{P}^N$, i.e., $(x_0 : \cdots : x_N), (y_0 : \cdots : y_N) \in (k^{N+1} - \{0\})/k^\times$, we require that the bihomogeneous polynomials $x_i y_j - x_j y_i$ vanish on it. The same schematic point p is the image of many k-valued points: take any field extension $k \supset \kappa_p$. However, vanishing of this equation does not depend on k.

We can also look at the equations in affine charts and rewrite the equations in (non-homogeneous) coordinates, in which the bihomogeneous polynomial $x_i y_j - x_j y_i$ becomes $\frac{x_i}{x_j} - \frac{y_i}{y_j}$. This allows us to define the zero locus as a closed subscheme.

It remains to show that $\mathbb{P}^N$ is universally closed, i.e., for any scheme S, the morphism $\pi : \mathbb{P}^N \times S \to S$ is closed. Without loss of generality, $S = \operatorname{Spec} R$ is affine, so $\mathbb{P}^N \times S = \mathbb{P}^N_R$. Let $X \subset \mathbb{P}^N_R$ be a closed subscheme.

We claim X is the zero locus of a family $(f_\alpha \in R[x_0, \ldots, x_N])_\alpha$ of homogeneous polynomials. (As before, "zero locus" is defined in terms of k-valued points for various fields k.) In affine charts, since X is closed, $X \cap \operatorname{Spec} R[\frac{x_0}{x_i}, \ldots, \frac{x_N}{x_i}]$ is the zero locus of some polynomials $g_\beta \in R[\frac{x_0}{x_i}, \ldots, \frac{x_N}{x_i}]$. Taking $f_\beta = x^{\deg(g_\beta)+1} g_\beta$ for all β and all charts yields the claim.

What is $\pi(X)$ in terms of equations? Given an algebraically closed field k and a geometric point $s : \operatorname{Spec} k \to \operatorname{Spec} R$ corresponding to a ring map $\varphi : R \to k$ and lying over a schematic point $p \in \operatorname{Spec} R$, the fiber $X_s := X \times_{\operatorname{Spec} R} \operatorname{Spec} k$ is given by $X \cap \pi^{-1}(p)$. This is empty if and only if the system of equations $\varphi(f_\alpha)(x_0 : \cdots : x_N) = 0$ has no solutions.

We'll finish the proof next time.

29 2015-04-08: Projective space is proper, continued

Continuing from last time, if $\mathfrak{m}$ is the prime ideal corresponding to p, then by the Nullstellensatz, X_s is empty if and only if $\mathfrak{m}^d \subset (\varphi(f_\alpha))$ for some $d \in \mathbb{N}$. The space of degree d polynomials in $(\varphi(f_\alpha))$ is spanned by polynomials of the form $\varphi(f_\alpha) \cdot g$ with $\deg(g) = d - \deg(f_\alpha)$.

Put $M := \dim(\mathfrak{m}^d)$. There exist $\alpha_1, \ldots, \alpha_M$ and $g_{\alpha_1}, \ldots, g_{\alpha_M}$ such that $(\varphi(f_{\alpha_i}) \cdot g_{\alpha_i})_{i=1}^M$ are linearly independent vectors in $\mathfrak{m}^d$. The condition is equivalent to a certain $M \times M$ determinant being nonzero.

This determinant makes sense in R, then we apply φ and see whether it's nonzero (i.e., whether this determinant is in the kernel of φ), which is a Zariski-open condition. So $\pi(X)$ has open complement, and hence is closed. $\square$

30 2015-04-10: Valuative criteria

[I was at a conference and missed this class. The subject of this class was the valuative criteria for separatedness and properness.]

Proposition 30.1 (Valuative criteria). *Let $f : X \to Y$ be a morphism of schemes such that X is Noetherian. Then f is separated iff for any valuation ring A with field of fractions K, any diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\varphi"] & Y \end{tikzcd}$$

admits at most one morphism φ, and proper iff any such diagram admits exactly one morphism φ.

31 2015-04-13: Vector bundles

Let M be a complex manifold. A vector bundle on M consists of a manifold E (the *total space*), a map $\pi : E \to M$, and a vector space structure on $\pi^{-1}(x)$ for each $x \in M$, such that E is locally trivial (i.e., there is a cover $M = \bigcup_\alpha U_\alpha$ such that $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}^d$ compatibly with π and the vector space structures).

More explicitly, fix a trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times \mathbb{C}^d$. On $U_\alpha \cap U_\beta$, we require

$$\varphi_\beta \circ \varphi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^d \to (U_\alpha \cap U_\beta) \times \mathbb{C}^d$$

to be a diffeomorphism compatible with projection to $U_\alpha \cap U_\beta$ and the vector space structure on $\mathbb{C}^d$. In other words, $\varphi_\beta \circ \varphi_\alpha^{-1} \in \mathrm{GL}(d, C^\infty(U_\alpha \cap U_\beta))$. These transition functions must satisfy the cocycle condition.

To a vector bundle $\pi : E \to M$, we assign a sheaf $\mathscr{E}$ of sections of π: for any open $U \subset M$,

$$\Gamma(U, \mathscr{E}) = \left\{ s : U \to E \;\middle|\; \pi \circ s = \mathrm{id} \right\}.$$

In fancier language, $\mathscr{E}$ is a sheaf of modules over the sheaf of smooth functions $\mathcal{O}_M$ on M.

Definition 31.1. Let X be a topological space. Let $\mathcal{O}_X$ be a sheaf of rings on X. A *module over $\mathcal{O}_X$* is a sheaf of abelian groups $\mathscr{E}$ together with a structure of a $\Gamma(U, \mathcal{O}_X)$-module on each $\Gamma(U, \mathscr{E})$, compatible with restriction.

Since a vector bundle $E \to M$ is locally trivial on some cover $M = \bigcup_\alpha U_\alpha$, its sheaf of sections $\mathscr{E}$ is a locally free $\mathcal{O}_M$-module of finite rank, i.e., $\mathscr{E}|_{U_\alpha} \cong (\mathcal{O}_M|_{U_\alpha})^d$ as $\mathcal{O}_M|_{U_\alpha}$-modules.

Theorem 31.2. *This gives an equivalence of categories*

$$\{\textit{vector bundles on } M\} \longleftrightarrow \left\{ \begin{array}{c} \textit{locally free } \mathcal{O}_M\textit{-modules} \\ \textit{of finite rank} \end{array} \right\}.$$

To prove this, it suffices to show that transition functions for locally free sheaves are also in $\mathrm{GL}(d, C^\infty(U_\alpha \cap U_\beta))$.

32 2015-04-15: Locally free sheaves

Definition 32.1. Let $(M, \mathcal{O}_M)$ be a ringed space. Let $\mathscr{E}$ be a sheaf of $\mathcal{O}_M$-modules. We say $\mathscr{E}$ is *locally free* iff every point of M has a neighborhood U such that $\mathscr{E}|_U \cong (\mathcal{O}_M|_U)^d$, the free $\mathcal{O}_M|_U$-module of rank $d \geq 0$.

Locally free $\mathcal{O}_M$-modules form a full subcategory of the category of $\mathcal{O}_M$-modules (where morphisms are morphisms of sheaves that respect the action of $\mathcal{O}_M$).

Exercise 32.2. $\mathrm{Hom}_{\mathcal{O}_M}(\mathcal{O}_M^d, \mathcal{O}_M^{d'}) = \mathrm{Mat}_{d' \times d}(\Gamma(M, \mathcal{O}_M))$.

Exercise 32.3. Let $\mathscr{E}$ be a locally free sheaf. Then $\mathrm{Hom}_{\mathcal{O}_M}(\mathcal{O}_M, \mathscr{E}) = \Gamma(M, \mathscr{E})$.

This imply a description of locally free sheaves via transition maps: fix an open cover $M = \bigcup_\alpha U_\alpha$, and on each U_α fix $(\mathcal{O}_M|_{U_\alpha})^{d_\alpha}$. If $U_\alpha \cap U_\beta \neq \varnothing$, then $d_\alpha = d_\beta =: d$, and we are given a transition map in $\mathrm{GL}(d, \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_M))$, satisfying the cocycle condition on triple intersections. Conversely, such data glues to form a locally free sheaf.

Exercise 32.4. Describe morphisms of locally free sheaves in charts.

We have an equivalence of categories

$$\{\mathbb{C}\text{-vector bundles}\} \xrightarrow{\simeq} \{\text{locally free sheaves of } C^\infty_{\mathbb{C}}\text{-modules}\}$$
$$E \mapsto \mathscr{E} = \text{sheaf of sections of } E.$$

What is the inverse functor? (It's not the espace étalé — that's too big.) As a set, one can recover the fibers as $E_x = (\mathscr{E}_x) \otimes_{(\mathcal{O}_M)_x} \mathbb{C}$, where we view $\mathbb{C}$ as an $(\mathcal{O}_M)_x$-module via the evaluation homomorphism $f \mapsto f(x) : (\mathcal{O}_M)_x \to \mathbb{C}$.

32.1 The algebraic setting

Definition 32.5. Let X be a scheme. A *vector bundle* on X is a morphism of schemes $\pi : E \to X$ such that every point of X has a Zariski-open neighborhood U such that $\pi^{-1}(U) \cong U \times \mathbb{A}^d = \mathbb{A}^d_U$ with transition maps linear in the coordinates of $\mathbb{A}^d$, together with the structure of a "family of vector spaces" on E (e.g., an addition map $E \times_X E \to E$).

In this setting as well, there is an equivalence of categories between locally free $\mathcal{O}_X$-modules and vector bundles on X.

33 2015-04-17: Examples of line bundles

Example 33.1 (Trivial line bundle). Let X be a scheme. Then endomorphisms of the trivial line bundle $X \times \mathbb{A}^1 \to X$ correspond to global functions $f \in \Gamma(X, \mathcal{O}_X)$, acting by $(x, t) \mapsto (x, f(x)t)$.

Example 33.2 (Tautological line bundle). Let $E = \{(t, \ell) \in \mathbb{A}^{n+1} \times \mathbb{P}^n : t \in \ell\} \subset \mathbb{A}^{n+1} \times \mathbb{P}^n$, where we view $\mathbb{P}^n$ as the space of lines through the origin in $\mathbb{A}^{n+1}$. The projection $(t, \ell) \mapsto \ell : E \to \mathbb{P}^n$ is a line bundle, called the *tautological line bundle*. Its sheaf of sections is

$$U \mapsto \{(t_0, \ldots, t_n) \in \Gamma(U, \mathcal{O}_{\mathbb{P}^n}) : x_i t_j - x_j t_i = 0\},$$

the sheaf of "functions of homogeneous degree -1 on U".

34 2015-04-24: Picard groups

Earlier, we discussed the operations of dual and tensor product on line bundles, which always produce another line bundle. Moreover, $L \otimes L^\vee = \mathcal{O}_X$ for any line bundle L. Hence, the set of isomorphism classes of line bundles on X is a group under tensor product, called the *Picard group* $\operatorname{Pic} X$.

Example 34.1. $\operatorname{Pic} \mathbb{P}^n \cong \mathbb{Z}$ is freely generated by $\mathcal{O}_{\mathbb{P}^n}(1)$.

Example 34.2. Let $X = \operatorname{Spec} k$, where k is a field (not necessarily algebraically closed). A sheaf of $\mathcal{O}_X$-modules is the same as a k-vector space. The locally free $\mathcal{O}_X$-modules of finite rank correspond to finite-dimensional k-vector spaces. Finally, $\operatorname{Pic} X$ is the trivial group.

Example 34.3. Let $X = \operatorname{Spec} R$, where R is a Dedekind domain. Then $\operatorname{Pic} X \cong \operatorname{Cl} R$.

How to assign a vector bundle to a sheaf? Its fiber over a point x is the stalk $\mathscr{E}_x$, which is a free $\mathcal{O}_{X,x}$-module.

35 2015-04-04: Vector fields and derivations

Let R be a commutative ring. Vector fields on $\operatorname{Spec} R$ are given by k-linear *derivations* $R \to R$, i.e., additive maps $\delta : R \to R$ such that $\delta(ab) = a\delta(b) + b\delta(a)$ and $\delta|_k = 0$. The R-module of all derivations $R \to R$ is denoted $\operatorname{Der}(R, R)$. Note that

$$\operatorname{Der}(R, R) = \operatorname{Hom}_R(\Omega^1_{R/k}, R).$$

Caution 35.1. Generally speaking, $\Omega^1_{R/k} \neq \operatorname{Hom}_R(\operatorname{Der}(R, R), R)$ (but this is true if R/k is smooth).

What happens if we pass from R to $R[f^{-1}]$? We have a natural map

$$\Omega^1_{R/k} \otimes_R R[f^{-1}] = \Omega^1_{R/k}[f^{-1}] \to \Omega^1_{R[f^{-1}]/k}$$

given by sending af^{-m} to $d(af^{-m})$.

Proposition 35.2. *This is an isomorphism.*

Moreover,

$$\operatorname{Hom}_{R[f^{-1}]}(\Omega^1_{R[f^{-1}]/k}, R[f^{-1}]) = \operatorname{Der}(R[f^{-1}], R[f^{-1}]) = \operatorname{Der}(R, R[f^{-1}]) = \operatorname{Hom}_R(\Omega^1_{R/k}, R[f^{-1}]) = \operatorname{Hom}_R(\Omega^1_{R/}$$

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