Math 764. Homework 8
Due Wednesday, April 15th

1. Let X be a variety. A sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ is said to be *radical* if for every open $U \subset X$, $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ is a radical ideal.

Show that $\mathcal{I}$ is radical if and only if $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ is radical for every point $x \in X$.

2. Let X be a variety. Show that we have an inclusion-reversing correspondence between closed subvarieties $Y \subset X$ and quasicoherent radical ideal sheaves $\mathcal{I} \subset \mathcal{O}_X$.

3. (From the video) Let $f : X \to Y$ be a morphism of affine varieties, so that f corresponds to a homomorphism of k-algebras $f^* : k[Y] \to k[X]$. Show that the direct image f_* on quasi-coherent sheaves corresponds to the restriction of scalars on modules under the equivalence between quasicoherent sheaves on an affine variety and modules over its coordinate ring.

4. (Also from the video) Let $f : X \to Y$ be a map of varieties that is quasi-compact: the preimage of a quasi-compact open subset is quasi-compact. Prove that the direct image f_* preserves quasi-coherence.

5. Let X be a topological space, and let $\mathcal{F}$, $\mathcal{G}$ be two sheaves of X (let's say they are sheaves of sets, although the claim holds for sheaves in any category). Define a pre-sheaf of sets $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$ on X by

$$\mathcal{H}om_X(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_U(\mathcal{F}|_U, \mathcal{G}|_U),$$

where on the right we have the set of morphisms of sheaves on U.

Show that $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$ is in fact a sheaf: *the sheaf of morphisms* between $\mathcal{F}$ and $\mathcal{G}$. (Informally, the claim is that morphisms of sheaves can be constructed locally.)

6. Let now X be a variety and suppose that $\mathcal{F}$, $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules. We define the sheaf of homomorphisms of $\mathcal{O}_X$-modules by the same formula as in the previous problem:

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where on the right we have the set of morphisms of $\mathcal{O}_U$-modules. Note that the sheaf of homomorphisms is naturally a $\mathcal{O}_X$-module.

Prove that if $\mathcal{G}$ is quasicoherent and $\mathcal{F}$ is *coherent*, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, and that for any point x, we have a natural isomorphism of stalks:

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x).$$

(Side question: what goes wrong if $\mathcal{F}$ is only quasi-coherent?)