Math 764. Homework 7
Due Wednesday, April 8th

Let X be a topological space.

1. (**Restriction of a sheaf to an open set.**) Let $U \subset X$ be an open subset. For any (pre)sheaf $\mathcal{F}$ on X, define its restriction $\mathcal{F}|_U$ to be the (pre)sheaf on U with the same spaces of sections:

$$\mathcal{F}|_U(V) = \mathcal{F}(V) \qquad (V \subset U \subset X).$$

Verify that if $\mathcal{F}$ is a sheaf, then so is $\mathcal{F}|_U$, and that the functor $\mathcal{F} \mapsto \mathcal{F}|_U$ preserves monomorphisms, epimorphisms, and products. (On sheaves of abelian groups, this functor would be exact.)

2. (**Extension by zero.**) Let $U \subset X$ be an open subset. Given a sheaf of abelian groups $\mathcal{F}$ on U, the extension by zero $j_!\mathcal{F}$ of $\mathcal{F}$ (here j is the embedding $U \hookrightarrow X$) is the sheaf on X that is the sheafification of the presheaf $\mathcal{G}$ such that

$$\mathcal{G}(V) = \begin{cases} \mathcal{F}(V), & V \subset U \\ 0, & V \not\subset U. \end{cases}$$

Is the sheafification necessary in this definition? (Or maybe $\mathcal{G}$ is a sheaf automatically?)

3. Describe the stalks of $j_!\mathcal{F}$ over all points of X (and, if you want, the étalé space of $j_!\mathcal{F}$).

4. Verify that $j_!$ is the left adjoint of the restriction functor from X to U: that is, for any sheaf of abelian groups $\mathcal{G}$ on X, there exists a natural isomorphism

$$\mathrm{Hom}_U(\mathcal{F}, \mathcal{G}|_U) \simeq \mathrm{Hom}_X(j_!\mathcal{F}, \mathcal{G}).$$

Side question (not part of the homework): What changes if we consider the version of extension by zero for sheaves of sets ('the extension by the empty set')?

5. Fix a point $x \in X$, and consider the functor $\mathcal{F} \mapsto \mathcal{F}_x$ from the category of sheaves of sets on X to the category of sets. Show that this functor admits a right adjoint and describe it.

6. (**Extension by** $*$) As before, suppose $U \subset X$ is open and $\mathcal{F}$ is a sheaf of abelian groups on U. Define its extension by $*$ (also known as the push-forward for open embedding) $j_*\mathcal{F}$ by

$$j_*(\mathcal{F})(V) = \mathcal{F}(U \cap V).$$

Verify that it is a sheaf, and try to find its stalks (the "try" part means figure out as much as you can about its stalks.)

7. Show that j_* is the right adjoint to restriction from X to U.