Math 764. Homework 5

Due Wednesday, March 25th

1. Let X be a curve of genus one. Show that X is parallelizable (i.e., its tangent bundle is trivial) without using explicit representation of X as a planar cubic.

2. Let $f : X \to Y$ be a morphism between smooth varieties. Show that the derivative of f induces a map between vector bundles TX and f^*TY (the pullback of TY via f).

3. Let $f : X \to Y$ be a non-constant morphism of smooth curves. For every point $x \in X$, let t be a local parameter at its image $f(x) \in Y$; the order of zero of the composition $t \circ f$ at x is called the *ramification index* at x. If the ramification index is greater than one, we call x a *ramification point* of f (and $f(x)$ a *branch point*).

Show that f has a ramification at x if and only if its derivative at x

$$f'(x) : T_x X \to T_{f(x)} Y$$

is zero.

4. Continuing in the situation of the previous problem, we say that f is *separated* if the induced field extension $k(X) \supset k(Y)$ (which is automatically finite because the two fields have the same transcendence degree) is separated.

Show that f is separated if and only if it has finitely many ramification points.

(Remark: this can be generalized to higher dimension: if $f : X \to Y$ is a dominant map between smooth varieties of the same dimension, its ramification locus is the set of points where its derivative fails to be invertible. The ramification locus is always a closed subset of X; it is a proper subset if and only if f is separable).

5. Let X be a curve, and E a vector bundle on X. Recall that the top exterior power $L := \wedge^{\mathrm{rk}(E)} E$ is a line bundle.

The *degree* of the vector bundle E is defined to be the degree of L: $\deg(E) = \deg(L)$.

Given two vector bundles E_1, E_2 on X, find formulas for degrees and ranks of vector bundles $\deg(E_1^\vee)$, $\deg(E_1 \oplus E_2)$, and $\deg(E_1 \otimes E_2)$ in terms of degrees and ranks of E_1 and E_2.

6. Let X_1 and X_2 be projective varieties, and let E_1 and E_2 be vector bundles on X_1 and X_2, respectively. The *external tensor product* $E_1 \boxtimes E_2$ is defined by

$$E_1 \boxtimes E_2 = p_1^* E_1 \otimes p_2^* E_2;$$

here $p_{1,2} : X_1 \times X_2 \to X_{1,2}$ are the two projections. Informally, $E_1 \boxtimes E_2$ is the vector bundle whose fiber over $(x_1, x_2) \in X_1 \times X_2$ is $(E_1)_{x_1} \otimes (E_2)_{x_2}$.

Prove a version of the *Künneth formula* that provides a natural isomorphism

$$\Gamma(X_1 \times X_2, E_1 \boxtimes E_2) \simeq \Gamma(X_1, E_1) \otimes \Gamma(X_2, E_2).$$

(Projectivity of varieties is not really required here, but it might make the argument simpler.)