Math 764. Homework 9

Due Friday, April 21st

1. Let X be a singular cubic in $\mathbb{P}^2$, given (in non-homogeneous coordinates) either by $y^2 = x^3 + x^2$ (nodal cubic) or by $y^2 = x^3$ (cuspidal cubic). Compute the class group of Cartier divisors on X.

2. Let X and Y be schemes over some base scheme S. For any map $f : X \to Y$, use the functoriality of the module of Kähler differentials to construct a morphism $f^*\Omega_{Y/S} \to \Omega_{X/S}$ and verify that $\Omega_{X/Y} = \operatorname{coker}(f^*\Omega_{Y/S} \to \Omega_{X/S})$.

3. Suppose now that X and Y be schemes over an algebraically closed field k. A morphism $f : X \to Y$ is *unramified* if $\Omega_{X/Y} = 0$. Show that this is equivalent to the following condition: given $D = \mathit{Spec}\, k[\epsilon]/\epsilon^2$, the map f induces an injection $\operatorname{Maps}(D, X) \to \operatorname{Maps}(D, Y)$.

4. Let us compute the algebraic de Rham cohomology of the affine space. Put $X = \mathit{Spec}\, R$, $R = k[t_1, \dots, t_n]$. Since X is a smooth k-scheme, $\Omega^1_R = \Omega_{R/k}$ is a locally free R-module. Denote by $\Omega^\bullet_R$ the exterior algebra of Ω^1_R, so that $\Omega^i_R = \bigwedge^i \Omega^1_R$. Define the de Rham differential $d : \Omega^i_R \to \Omega^{i+1}_R$ by starting with the Kähler differential $d : R \to \Omega^1_R$ and then extending it by the graded Leibniz rule:

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^i \omega_1 \wedge d(\omega_2), \qquad \omega_1 \in \Omega^i_R.$$

Compute the cohomology of the complex $\Omega^\bullet_R$ equipped with the differential d. The answer will depend on the characteristic of k.

5. Let X be a Noetherian scheme.Let $K(X)$ be the *K-group of X*: it is generated by elements $[F]$ for each coherent sheaf F with relations $[F] = [F_1] + [F_2]$ whenever there is a short exact sequence

$$0 \to F_1 \to F_2 \to F_3 \to 0.$$

Prove that $K(\mathbb{A}^n) = \mathbb{Z}$. (This is much easier if you know Hilbert's Syzygy Theorem.)

6. Let X be a smooth curve over an algebraically closed field. Show that $K(X)$ is generated by $[L]$ for line bundles L.

7. Let X be a smooth curve over an algebraically closed field. Show that $K(X)$ is isomorphic to $\mathbb{Z} \oplus \operatorname{Pic}(X)$. (If this problem is too hard, look at Hartshorne's II.6.11 for a step-by-step approach.)