Math 764. Homework 8
Due Friday, April 7th

1. (Hartshorne, II.4.4) Fix a Noetherian scheme S, let X and Y be schemes of finite type and separated over S, and let $f : X \to Y$ be a morphism of S-schemes. Suppose that $Z \subset X$ be a closed subscheme that is proper over S. Show that $f(Z) \subset Y$ is closed.

2. In the setting of the previous problem, show that if we consider $f(Z)$ as a closed subscheme (its ideal of functions consists of all functions whose composition with f is zero), then f induces a proper map fro Z to $f(Z)$.

(Galois descent, inspired by Hartshorne II.4.7) Let F/k be a finite Galois extension of fields. The Galois group $G := Gal(F/k)$ acts on the scheme $Spec(F)$. Given any k-scheme X, we let $X_F := Spec(F) \times_{Spec(k)} X$ be its extension of scalars; the group G acts on X_F in a way compatible with its action on $Spec(F)$ (i.e., this action is 'semilinear').

3. Show that X is affine if and only if X_F is affine.

4. Prove that this operation gives a fully faithful functor from the category of k-schemes into the category of F-schemes with a semi-linear action of G.

5. Suppose that Y is a separated F-scheme such that any finite subset of Y is contained in an affine open chart (this holds, for instance, if Y is quasi-projective). Then for any semi-linear action of G on Y, there exists a k-scheme X and an isomorphism $X_F \simeq Y$ that agrees with an action of G. (That is, the action of G gives a k-structure on the scheme Y.)

6. Suppose X is an $\mathbb{R}$-scheme such that $X_{\mathbb{C}} \simeq \mathbb{A}^1_{\mathbb{C}}$. Show that $X \simeq \mathbb{A}^1_{\mathbb{R}}$.

7. Suppose X is an $\mathbb{R}$-scheme such that $X_{\mathbb{C}} \simeq \mathbb{P}^1_{\mathbb{C}}$. Show that there are two possibilities for the isomorphism class of X.