Math 764. Homework 4
Due Friday, February 24th

1. Show that the following two definitions of quasi-separated-ness of a scheme S are equivalent:
(1) The intersection of any two quasi-compact open subsets of S is quasi-compact;
(2) There is a cover of S by affine open subsets whose (pairwise) intersections are quasi-compact.

2. In class, we gave the following definition: a scheme S is *integral* if it is irreducible and reduced. Show that this is equivalent to the definition from Vakil's notes: a scheme is integral if for any non-empty open $U \subset S$, $O_S(U)$ is a domain.

3. Let us call a scheme X *locally irreducible* if every point has an irreducible neighborhood. (Since a non-empty open subset of an irreducible space is irreducible, this implies that all smaller neighborhoods of this point are irreducible as well.) Prove or disprove the following claim: a scheme is irreducible if and only if it is connected and locally irreducible.

4. Show that a locally Noetherian scheme is quasi-separated.

5. Show that the following two definitions of a Noetherian scheme X are equivalent:
(1) X is a finite union of open affine sets, each of which is the spectrum of a Noetherian ring;
(2) X is quasi-compact and locally Noetherian.

6. Show that any Noetherian scheme X is a disjoint union of finitely many connected open subsets (the *connected components* of X.) (A problem from the last homework shows that things might go wrong if we do not assume that X is Noetherian.)

7. A locally closed subscheme $X \subset Y$ is defined as a closed subscheme of an open subscheme of Y. Accordingly, a locally closed embedding is a composition of a closed embedding followed by an open embedding (in this order). In principle, one can try to reverse the order, and consider open subschemes of closed subschemes of Y. Does this yield an equivalent definition?

Remark. The difficulty of such questions (and, sometimes, the answer to them) depends on the class of schemes one works with: often, very mild assumptions (such as, say, quasicompactness) would make the question easy. A complete answer to this problem would include both the mild assumptions that would make the two versions equivalent, and a description of what happens for general schemes.