Math 763. Homework 6

Due Thursday, November 7th

1. Show that $\mathbb{P}^1 \times \mathbb{A}^1$, $\mathbb{P}^2 - \{*\}$, and $\mathbb{A}^2 - \{*\}$ are non-isomorphic varieties that are neither projective nor affine. (Here $*$ is a single point.)

2. *Veronese embedding:* Fix integers $n, d > 0$. The Veronese morphism $\nu : \mathbb{P}^n \to \mathbb{P}^N$ sends $(x_0 : \cdots : x_n)$ to the point whose homogeneous coordinates are all possible degree d monomials in x_i's (so $N = \binom{n+k}{k} - 1$). It is clear that ν is a regular map. Prove that $\nu(\mathbb{P}^n) \subset \mathbb{P}^N$ is closed by writing a set of equations on it (rather than by using the general theorem).

3. Show that $\nu : \mathbb{P}^n \to \mathbb{P}^N$ is a closed embedding.

4. Show that for every non-constant homogeneous polynomial f, the 'principal open set' $\mathbb{P}^n - V(f)$ is affine. (Hint: use the Veronese embedding.)

5. Suppose a projective variety $X \subset \mathbb{P}^n$ of pure dimension k is a set-theoretic complete intersection: it is a zero locus of $n - k$ homogeneous polynomials. Prove that X meets any non-empty closed subvariety $Y \subset \mathbb{P}^n$ of dimension $n - k$. (Hint: consider the affine cone?)

6. *Discriminants:* Let us show how to introduce the discriminant of a polynomial without writing any formulas. Fix n, and consider, for any point

$$a = (a_0 : \cdots : a_n) \in \mathbb{P}^n,$$

the polynomial

$$p(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n.$$

Denote by $X \subset \mathbb{P}^n$ the set of all points a such that the polynomial $p(x, y)$ has less than n zeroes on $\mathbb{P}^1$. In other words, X is the set of polynomials that have at least one multiple zero.

Prove that $X = V(D)$ for a homogeneous irreducible polynomial $D \in k[a_0, \ldots, a_n]$ (the discriminant). (You can look up an explicit formula for D on Wikipedia, but please do not use the formula. Instead try showing that $X \subset \mathbb{P}^n$ is an irreducible hypersurface.)