Math 763. Homework 2

Due Thursday, September 26th

In these problems (and everywhere else in the class), the ground field, which is denoted by k, is assumed to be algebraically closed.

1. Prove that a subspace of a noetherian topological space is noetherian in the induced topology.

2. Show that a topological space is noetherian if and only if all of its open subsets are quasi-compact.

3. Let $S \subseteq k$ be an infinite subset (for instance, $S = \mathbb{Z} \subset k = \mathbb{C}$.) Show that the Zariski closure of the set

$$X = \{(s, s^2) | s \in S\} \subset \mathbb{A}^2$$

is the parabola

$$Y = \{(a, a^2) | a \in k\} \subset \mathbb{A}^2.$$

4. Show that an algebraic set X is connected if and only if the algebra $k[X]$ has no idempotents other than 0 and 1. (Recall that f is an idempotent if $f^2 = f$.)

5. Let

$$X = Z(x^2 - z^2 + y, yz - y) \subset \mathbb{A}^3,$$

and assume that $char(k) \neq 2$. Show that X is a union of three irreducible components. Describe them and find their prime ideals.

6. Let $f : X \to Y$ be a regular map; consider the induced map $f^* : k[Y] \to k[X]$. Given an ideal I of $k[Y]$, consider the ideal $k[X] \cdot f^*(I)$ generated by its image in $k[X]$. Describe the closed subset

$$V(k[X] \cdot f^*(I)) \subset X$$

in terms of

$$V(I) \subset Y.$$

7. Let $f : X \to Y$ be a regular map; consider the induced map $f^* : k[Y] \to k[X]$. Given an ideal I of $k[X]$, its preimage $(f^*)^{-1}(I)$ is an ideal in $k[Y]$. Describe the corresponding closed subset

$$V((f^*)^{-1}(I)) \subset Y$$

in terms of

$$V(I) \subset X.$$

You may assume that I is radical.

8. Let $f : X \to Y$ be a regular map between algebraic sets. Suppose that the image of f is dense in the Zariski topology of Y. (Such maps f are called *dominant*.) Prove that if X is irreducible, then so is Y.

Remark: There are two proofs: geometric and algebraic.