Math 222 Week 4 Homework - Solutions

I.2.3: The key idea here is what they mean by the inequality in each case. In a, it is inequalities of numbers, in b it is inequalities of functions, and in c it is ambiguous. None of the inequalities are true.

By direct computation, the integral in a is $\left[x - \frac{x^3}{3}\right]_2^4 = (4 - \frac{64}{3}) - (2 - \frac{8}{3}) = -\frac{50}{3}$. Alternatively, we could use the fact that $1 - x^2$ is always negative on the interval $[2, 4]$, so the integral in a must be negative, so the inequality in a is false.

For part b, note that we are integrating $(1 - x^2)$ with respect to t, from 2 to 4. Therefore, the integral is $(1 - x^2)(4 - 2) = 2(1 - x^2)$. As a function, this is not always positive, so the inequality in b is false.

For part c, we are finding $\int (1 + x^2)dx = x + \frac{x^3}{3} + C$. However, this is not a well-defined function, as it depends on what C we pick. Therefore, it doesn't make sense to ask if $x + \frac{x^3}{3} + C > 0$, so this inequality can't be true.

I.7.5: Let $I_n = \int (\sin x)^n dx$. From equation 6.3, we have the formula:

$$I_n = \frac{-1}{n} \sin^{n-1}(x)\cos(x) + \frac{n-1}{n} I_{n-2}$$

We need to use this formula to compute $\int \sin^2(x)dx$. Using the formula, we get:

$$\begin{aligned}
\int \sin^2(x)dx = I_2 &= \frac{-1}{2}\sin^1(x)\cos(x) + \frac{1}{2}I_0 \\
&= \frac{-1}{2}\sin(x)\cos(x) + \frac{1}{2}\int \sin^0(x)dx \\
&= \frac{-1}{2}\sin(x)\cos(x) + \frac{1}{2}x + C
\end{aligned}$$

Alternatively, we can integrate $\sin^2(x)$ using the half angle formula as follows:

$$\begin{aligned}
\int \sin^2(x)dx &= \int \frac{1}{2}(1 - \cos(2x))dx \\
&= \frac{1}{2}\int (1 - \cos(2x))dx \\
&= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C \\
&= \frac{1}{2}x - \frac{1}{2}\sin(x)\cos(x) + C
\end{aligned}$$

The last step was using the fact that $\sin(2x) = 2\sin(x)\cos(x)$. We get the same answer as before.

I.7.20: Let $I_n = \int \frac{dx}{(1+x^2)^n}$. If you look at the example in 6.4, you will find that as long as $n \neq 0$, the reduction formula still works. In the case where $n = 0$, this becomes $\int \frac{dx}{1}$, which is just $x + C$. Letting $n = -\frac{1}{2}$, we get:

$$\int \sqrt{1+x^2}dx = \int (1+x^2)^{1/2}dx = \int \frac{dx}{(1+x^2)^{-1/2}} = I_{-1/2}$$

Additionally, we have:

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{dx}{(1+x^2)^{1/2}} = I_{1/2}$$

So, using the reduction formula to compare $I_{1/2}$ and $I_{-1/2}$, we have:

$$\begin{aligned} &I_{1/2} = I_{-1/2+1} = \frac{1}{2(-1/2)} \frac{x}{(1+x^2)^{-1/2}} + \frac{2(-1/2)-1}{2(-1/2)} I_{-1/2} \\ &\to I_{1/2} = -x\sqrt{1+x^2} + 2I_{-1/2} \end{aligned}$$

I.7.21: We use integration by parts to integrate $\frac{1}{x}$. Let $F = \frac{1}{x}, G' = 1$. Then $F' = \frac{-1}{x^2}, G = x$. So, we get:

$$\begin{aligned} \int \frac{1}{x}dx &= FG - \int F'Gdx \\ &= \frac{1}{x}x - \int \frac{-1}{x^2}xdx \\ &= 1 + \int \frac{1}{x}dx \end{aligned}$$

Subtracting, we get $\int \frac{1}{x}dx - \int \frac{1}{x}dx = 1$. This is not wrong though! Remember that antiderivatives can be different up to a constant. So, let c_1 be the constant for the first integral, and c_2 be the constant for the second.

Then $\int \frac{1}{x}dx - \int \frac{1}{x}dx = c_1 - c_2$, a constant. It does not have to equal 0, so the equation from before makes sense. It just tells us that the two constants c_1, c_2 differ by 1.

I.9.15: We wish to find $\displaystyle\int \frac{e^x}{\sqrt{1+e^{2x}}}dx$.

First, let $u = e^x$. Then $x = \ln(u)$, so $dx = \frac{1}{u}du$. The integral therefore becomes:

$$\int \frac{u}{\sqrt{1+u^2}} \frac{1}{u}du = \int \frac{1}{\sqrt{1+u^2}}du$$

We now have to do another substitution to get rid fo the square root. Let $u = \tan(\theta)$, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $du = \sec^2(\theta)d\theta$. Because $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\sec(\theta) > 0$, so $\sqrt{1+u^2} = \sqrt{1+\tan^2(\theta)} = \sec(\theta)$. The integral becomes:

$$\begin{aligned} \int \frac{1}{\sqrt{1+u^2}}du &= \int \frac{\sec^2(\theta)}{\sec(\theta)}d\theta \\ &= \int \sec(\theta)d\theta \\ &= \ln|\sec(\theta) + \tan(\theta)| + C \end{aligned}$$

We know that $\tan(\theta) = u = e^x$, and $\sec(\theta) = \sqrt{1+u^2} = \sqrt{1+e^{2x}}$, so the final answer is:

$$\ln|\sqrt{1+e^{2x}} + e^x| + C$$

I.13.3: We want to find $\int \sqrt{1+x^2}dx$. To get rid of the square root, we use a rational substitution. Recall $U(t) = \frac{1}{2}(t+\frac{1}{t}), V(t) = \frac{1}{2}(t-\frac{1}{t})$. Also recall $U^2 = V^2+1$. We do the substitution $x = V, t \geq 1$. So, $dx = V'dt$, and our integral becomes:

$$\begin{aligned}
\int \sqrt{1+x^2}dx = \int \sqrt{1+V^2}V'dt &= \int UV'dt \\
&= \int \frac{1}{2}(t+\frac{1}{t})\frac{1}{2}(1+\frac{1}{t^2})dt \\
&= \frac{1}{4}\int t + \frac{2}{t} + \frac{1}{t^3}dt \\
&= \frac{1}{4}\left(\frac{t^2}{2} + 2\ln|t| - \frac{1}{2t^2}\right) + C \\
&= \frac{1}{8}t^2 - \frac{1}{8}t^{-2} + \frac{1}{2}\ln|t| + C
\end{aligned}$$

The easiest way to substitute back in the x is to use the facts (given in the book):

1. $t = U + V$
2. $\frac{1}{t} = U - V$

So, our expression from before becomes:

$$\begin{aligned}
&\frac{1}{8}(U+V)^2 - \frac{1}{8}(U-V)^2 + \frac{1}{2}\ln|U+V| + C \\
&= \frac{1}{8}(U^2+2UV+V^2) - \frac{1}{8}(U^2-2UV+V^2) + \frac{1}{2}\ln|U+V| + C \\
&= \frac{1}{2}UV + \frac{1}{2}\ln|U+V| + C
\end{aligned}$$

Finally, we can use the fact that $V = x$ and $U = \sqrt{1-V^2} = \sqrt{1-x^2}$. So the final answer is:

$$\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\ln|x+\sqrt{1-x^2}| + C$$

Alternatively, you could have used the substitution $x = \tan(\theta)$, and then figured out how to integrate $\sec^3(\theta)$.

I.13.4: We want to integrate $\dfrac{1}{\sqrt{2x-x^2}}$. We first complete the square.

We get $2x - x^2 = -(x^2-2x) = -((x-1)^2-1) = 1-(x-1)^2$. Our integral becomes $\displaystyle\int \frac{dx}{\sqrt{1-(x-1)^2}}$.

This looks like the derivative of $\arcsin(x-1)$. If you do the substitution, you find that this integrates to $\arcsin(x-1) + C$.

I.13.7: We want to integrate $\frac{1}{\sqrt{4-x^2}}$. This looks a lot like the derivative of arcsin. We factor out a 4 from the square root to get:

$$\begin{aligned}\int \frac{dx}{\sqrt{4-x^2}} &= \int \frac{dx}{2\sqrt{1-(x/2)^2}} \\ &= \int \frac{du}{\sqrt{1-u^2}}\end{aligned}$$

This last step was done using the substitution $u = \frac{x}{2}$. This integrates to $\arcsin(u) + C$, which equals $\arcsin(\frac{x}{2}) + C$.

I.13.15: We want to find $\int_1^{\sqrt{3}} \dfrac{dx}{x^2+1}$. Note that the function within the integral is the derivative of $\arctan(x)$, so the integral equals $\arctan(\sqrt{3}) - \arctan(1)$, where we take arctan with range in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

First, we calculate $\arctan(\sqrt{3})$. This equals the angle θ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\tan(\theta) = \sqrt{3}$. By basic trigonometry, we know this is when $\theta = \frac{\pi}{3}$, so $\arctan(\sqrt{3}) = \frac{\pi}{3}$. Next, we find $\arctan(1)$. This equals the angle θ such that $\tan(\theta) = 1$, so this equals $\frac{\pi}{4}$.

The final answer is $\frac{\pi}{3} - \frac{\pi}{4}$.

I.15.3: Let $u = x^2 - 1$. Then $du = 2xdx$, so the integral becomes:

$$\begin{aligned}\int \frac{x}{\sqrt{x^2-1}}dx &= \int \frac{\frac{1}{2}du}{\sqrt{u}} \\ &= \sqrt{u} + C \\ &= \sqrt{x^2-1} + C\end{aligned}$$

I.15.8: In this problem, we have a ratio of polynomials, so we use polynomial long division and partial fractions. First, we use polynomial long division since the numerator does not have smaller degree, getting:

$$\begin{array}{r|l} & x^2 \quad + 36 \\ \hline x^2 - 36 & x^4 \\ & -x^4 + 36x^2 \\ \hline & 36x^2 \\ & -36x^2 + 1296 \\ \hline & 1296 \end{array}$$

So, $\dfrac{x^4}{x^2-36} = x^2 + 36 + \dfrac{1296}{x^2-36}$. So we get:

$$\begin{aligned}\int \frac{x^4}{x^2-36}dx &= \int x^2 + 36 + \frac{1296}{x^2-36}dx \\ &= \frac{x^3}{3} + 36x + 1296 \int \frac{1}{x^2-36}dx\end{aligned}$$

We do partial fractions on $\frac{1}{x^2-36} = \frac{1}{(x-6)(x+6)}$. This must equal $\frac{A}{x-6}+\frac{B}{x+6}$ for some A, B. The heavyside method say $A = \frac{1}{x+6}\Big|_{x=6} = \frac{1}{12}$, and $B = \frac{1}{x-6}\Big|_{x=-6} = \frac{-1}{12}$.

So, the fraction equals $\frac{1}{12(x-6)} - \frac{1}{12(x+6)}$. Integrating this, we get $\frac{1}{12}\ln|x-6| - \frac{1}{12}\ln|x+6| + C$.

The final answer is $\dfrac{x^3}{3} + 36x + 1296\left(\frac{1}{12}\ln|x-6| - \frac{1}{12}\ln|x+6|\right) + C$.

I.15.34: We have to use partial fractions in both cases.

For the first, we know:

$$\frac{1}{x(x-1)(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{x-3}$$

We can use the heavyside method to calculate the coefficients. Using it, we get:

- $A = \dfrac{1}{(0-1)(0-2)(0-3)} = \dfrac{-1}{6}$
- $B = \dfrac{1}{(1)(1-2)(1-3)} = \dfrac{1}{2}$
- $C = \dfrac{1}{(2)(2-1)(2-3)} = \dfrac{-1}{2}$
- $D = \dfrac{1}{(3)(3-1)(3-2)} = \dfrac{1}{6}$

Integrating, we get $\dfrac{-1}{6}\ln|x| + \dfrac{1}{2}\ln|x-1| - \dfrac{1}{2}\ln|x-2| + \dfrac{1}{6}\ln|x-3| + E$.

For the second, we do almost exactly the same thing. The only thing that is potentially different are the coefficients. Using the heavyside method, they are calculated as follows:

- $A = \dfrac{(0)^3+1}{(0-1)(0-2)(0-3)} = \dfrac{-1}{6}$
- $B = \dfrac{(1)^3+1}{(1)(1-2)(1-3)} = 1$
- $C = \dfrac{(2)^3+1}{(2)(2-1)(2-3)} = \dfrac{-9}{2}$
- $D = \dfrac{(3)^3+1}{(3)(3-1)(3-2)} = \dfrac{28}{6} = \dfrac{14}{3}$

Integrating, we get $\dfrac{-1}{6}\ln|x| + \ln|x-1| - \dfrac{9}{2}\ln|x-2| + \dfrac{14}{3}\ln|x-3| + E$.

I.15.35: This is a ratio of polynomials, so we'd like to use partial fractions. To do that, we need to factor the denominator, $1+x+x^2+x^3$. As the book hints, this equals $1+x+x^2(1+x)$. Factoring out a $(1+x)$, this equals $(1+x)(1+x^2)$. Note that $1+x^2$ cannot be broken up any more, as we would have to use $\sqrt{-1}$. So, the two factors are $1+x, 1+x^2$.

Based on the method of partial fractions, we find:

$$\frac{1}{1+x+x^2+x^3} = \frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

We can't use the heavyside method because there is a quadratic factor. So, clearing the denominators instead, we get:

$$\begin{aligned} 1 &= A(x^2+1) + (Bx+C)(x+1) \\ &= Ax^2 + A + Bx^2 + Bx + Cx + C \\ &= x^2(A) + x(B+C) + A + C \end{aligned}$$

We must have $A = 0, B + C = 0, A + C = 1$. so, $A = 0, B = -1, C = 1$. Therefore, our integral is:

$$\begin{aligned} \int \frac{dx}{1+x+x^2+x^3} &= \int \frac{-x+1}{x^2+1} dx \\ &= -\int \frac{x}{x^2+1} dx + \frac{1}{x^2+1} \end{aligned}$$

To solve the first integral, we use $u = x^2 + 1$, so $du = 2xdx$. The integral becomes:

$$-\int \frac{\frac{1}{2}du}{u} = -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln|x^2+1|$$

The second integral is simply $\arctan(x)$, so the final answer is $-\frac{1}{2}\ln|x^2+1| + \arctan(x) + E$.

I.15.38: Once you've drawn the graph, note that it is symmetric around the x-axis. That is, you can reflect it to get the same graph. What this means is that any area above the x-axis (positive area) cancels out with the area below the x-axis, so the area under the curve is 0.

I.15.43: While the book claims that the function $F(x)$ is an anti-derivative for $f(x)$, it is not! An antiderivative needs to have the correct derivative on the entire interval $[0, 2]$. But $F(x)$ is not differentiable at the point $x = 1$. In fact, $F(x)$ is not even continuous here!. You get different values whether you are approaching 1 from the left or the right.

We have:
$\lim_{x \to 1^-} F(x) = \frac{1}{2}x^2 - x \Big|_{x=1} = -\frac{1}{2}$
$\lim_{x \to 1^+} F(x) = x - \frac{1}{2}x^2 \Big|_{x=1} = \frac{1}{2}$
Since $F(x)$ is not continuous at 1, it is not differentiable for 1, and so is not an antiderivative for $f(x)$ on the interval $[0, 2]$.

I.15.44: The issue here comes from the substitution. Remember that x has domain $[-1, 1]$, and $u = 1 - x^2$. Therefore, the book claims that $x = \sqrt{1-u}$. However, $\sqrt{1-u} \geq 0$ for all u, but x needs to be negative at times. This substitution does not work because it does not pass the horizontal line test on the interal $[-1, 1]$. When you graph $u = 1 - x^2$ on this interval, you will see that it fails this test.