For the last session, you solved this question from Summer 2012:

E0. Show that there exists an $\mathcal{N} \models \mathrm{PA}$ and an $a \in \mathcal{N} \smallsetminus \mathbb{N}$ so that a is definable in $\mathcal{N}$.

In a later qual, we asked a more sophisticated version:

E1. Let M be a model of PA that is not elementarily equivalent to $(\mathbb{N}, +, \cdot)$. Show that there is an infinite element of M that is definable.

What about nonstandard models of PA that *are* elementarily equivalent to $(\mathbb{N}, +, \cdot)$? In other words, what about nonstandard models of true arithmetic? This was never a qual problem, but for completeness:

E2. Let M be a model of True Arithmetic, i.e., the theory of $(\mathbb{N}, +, \cdot)$. Show that *no* infinite element of M is definable.

Incompleteness is intimately linked to computable inseparability.

E3. Consider the sets

$$A = \{\ulcorner\varphi\urcorner \colon \mathrm{PA} \vdash \varphi\},$$
$$B = \{\ulcorner\varphi\urcorner \colon \mathrm{PA} \vdash \neg\varphi\},$$

where $\ulcorner\varphi\urcorner$ is the Gödel code of the sentence φ. Show that A and B are computably inseparable. I.e., show that there is no computable set C such that $A \subseteq C$ and $B \cap C = \emptyset$.

The next question extends the fact that no computably enumerable, consistent extension of PA is complete.

E4. Let T_0 and T_1 be computably enumerable, consistent extensions of PA (although, $T_0 \cup T_1$ need not be consistent). Show that there is a sentence ψ that is independent of both T_0 and T_1.

That's enough about PA and its extensions. The next few problems are basic model theory.

E5. Call a model M "nice" iff for every $a, b \in M$, there is an automorphism of M that moves a to b. Let T be a theory in a countable language. Show that if T has a nice model of some infinite cardinality, then T has nice models of all infinite cardinalities.

E6. Let L be a language which includes a unary relation symbol R. Let φ be an L-sentence and Γ a set of L-sentences neither of which contains the symbol R. If Γ proves φ in the language L, must there be a deduction of φ from Γ in which R does not occur (i.e., in the language $L - \{R\}$)? If so, prove that there is always such a deduction; and if not, describe Γ and φ which provide a counterexample.

E7. Let L be the language containing one binary relation symbol. A graph is a symmetric irreflexive binary relation. It is n-colorable iff there is a map from its universe into n such that no two elements in the relation are assigned the same value.

(a) Show that there is a first order L-theory T whose models are exactly the 3-colorable graphs.

(b) Prove that T is not finitely axiomatizable.

We finish with a strange combinatorial problem.

E8. Prove that there is no family $\{A_\alpha : \alpha < \omega_1\} \subseteq \mathcal{P}(\omega)$ such that for all $\alpha < \beta$: $A_\beta \backslash A_\alpha$ is infinite and $|A_\alpha \backslash A_\beta| \leq 7$.