E1. Suppose L is a finite language, T is an L-theory, L' a language extending L, and θ is an L' sentence such that the models of T are exactly the reducts to L of the models of θ. Prove that T is computably axiomatizable.

E2. For any theory T let F be the set of all sentences in the language of T which are true in some finite model of T. Assume the language of T is recursive.

a. If T is finitely axiomatizable show F is c.e.

b. If T is decidable show F is c.e. Warning: The language of T might be infinite and even if its finite, T might not be finitely axiomatizable.

c. Give an example of a computably axiomatizable T in the language of pure equality such that F is not c.e.

E3. Let T be a finitely axiomatizable theory in a finite language $\mathcal{L}$. Assume that for each sentence θ of $\mathcal{L}$, either $T \cup \{\theta\}$ has a finite model or $T \cup \{\theta\}$ is inconsistent. Prove that the set of sentences of $\mathcal{L}$ which are provable from T is decidable.

E4. Let T be the theory of one infinite, coinfinite unary relation U. Show that T is decidable.

E5. Prove that the following are equivalent for any consistent first order theory T in a finite $\mathcal{L}$ which is not finitely axiomatizable:

a. T has a computable set of independent axioms.

b. T has a set of axioms which can be computably enumerated as $\theta_0, \theta_1, \theta_2, \ldots$, so that for every n, $\theta_{n+1} \to \theta_n$ is logically valid, but $\theta_n \to \theta_{n+1}$ is not.

c. Whenever $\rho_0, \rho_1, \rho_2, \ldots$ is any computable enumeration of some set of axioms for T, there exists a strictly increasing computable function $f : \omega \to \omega$ such that for every n:

$$\bigwedge_{k \le n} \rho_k \quad \text{does not imply} \quad \bigwedge_{k \le f(n)} \rho_k \ .$$

In (a): a set of axioms Σ for T is "*independent*" if no $\varphi \in \Sigma$ is provable from $\Sigma \backslash \{\varphi\}$.

E6. Let T be a consistent computably axiomatizable theory with only finitely many complete extensions in the same language. Show that T is decidable.

E7. Let T be a consistent c.e. axiomatizable extension of Peano Arithmetic. Show that there is an e such that the partial computable function φ_e is total, but T does not prove that φ_e is total.

E8. Let T be a computably axiomatizable complete theory in an infinite language. Prove that T has a computable set of axioms which is independent.

E9. Prove that any decidable consistent theory T which is closed under implication has a decidable complete consistent extension, $T' \supseteq T$, in the same language.