Linear algebra.
11/18/20

Linear algebra is a consistently popular topic in math competitions. I plan to start by discussing the example problems below, and then you'd work on the rest of the problems.

Examples (topics:rank, normal form).

1. Let A be a 5×10 matrix with real entries, and let A' denote its transpose (so A' is a 10×5 matrix, and the ijth entry of A' is the jith entry of A). Suppose every 5×1 matrix with real entries (i.e. column vector in 5 dimensions) can be written in the form Au where u is a 10×1 matrix with real entries. Prove that every 5×1 matrix with real entries can be written in the form $AA'v$ where v is a 5×1 matrix with real entries.

2. Suppose A, B are two-by-two matrices satisfying $AB = BA$. Prove that there exists two numbers a, b, not both equal to zero, such that $aA + bB$ is a scalar matrix.

More problems.

3. In a town with n people, m clubs have been formed. Every club has an odd number of members, and every two clubs have an even number of members in common. Prove that $m \leq n$.

4. Let n and k be positive integers. Say that a permutation σ of $\{1, 2, ..., n\}$ is k-limited if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of all k-limited permutations of $\{1, 2, \ldots, n\}$ is odd if and only if $n \equiv 0$ or $1 \mod 2k + 1$.

(There is probably a way to solve this problem directly... but can you figure out what it has to do with linear algebra?)

5. (Putnam 2006) Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are vertices of a hypercube in $\mathbb{R}^n$.) Given a vector subspace V of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subset \mathbb{R}^n$ of dimension k, of $Z(V)$.

6. Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m + n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

7. Let A and B be two $n \times n$ matrices. Show that $\mathrm{rk}(A) + \mathrm{rk}(B) \leq n$ if and only if there exists and invertible $n \times n$ matrix X such that $AXB = 0$.

Even more problems (to think about later on your own, or in case we magically have too much time on our hands):

8. Let n be a positive integer, let A, B be square symmetric $n \times n$ matrices with real entries (so if a_{ij} are the entries of A, the a_{ij} are real numbers and $a_{ij} = a_{ji}$). Suppose there are $n \times n$ matrices X, Y (with complex entries) such that

$$\det(AX + BY) \neq 0.$$

Prove that

$$\det(A^2 + B^2) \neq 0$$

(det indicates the determinant).

9. (I really hate this one) Let I denote the 2×2 identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and let

$$M = \begin{pmatrix} I & A \\ B & C \end{pmatrix}, N = \begin{pmatrix} I & B \\ A & C \end{pmatrix},$$

where A, B, C are arbitrary 2×2 matrices with entries in $\mathbb{R}$, the real numbers. Thus M and N are 4×4 matrices with entries in $\mathbb{R}$. Is it true that M is invertible (i.e. there is a 4×4 matrix X such that $MX = XM =$ the identity matrix) implies N is invertible? Justify your answer.

Here is my cheatsheet: a few useful facts and concepts from linear algebra (and I am sure I am forgetting some). Sound familiar?

- Determinants. They...
  - determine whether (a) a matrix is invertible, (b) its rows are linearly independent, (c) its columns are linearly independent;
  - can be expanded recursively by row or by column;
  - can be expressed as a sum over permutations;
  - change in a controllable way under row and column manipulations;
  - multiply when matrices get multiplied.
- Matrices:
  - Rank can be determined by columns and by rows.
  - Eigenvalues, eigevectors, characteristic polynomial, the Cayley-Hamilton Theorem.
- Vector spaces:
  - Basis, dimension, linear (in)dependence, and span.
  - Change of basis.