Linear algebra.
11/11/20 and 11/18/20

Linear algebra is a consistently popular topic in math competitions. I plan to start by discussing the example problems below, and then give you time to work on the other problems. (Or we can do something else - we'll see how it goes!)

Examples.

1. (Putnam 2014) Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is $1/\min(i,j)$ for $1 \leq i,j \leq n$. Compute $\det(A)$.

2. (Putnam 2003) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

3. (Putnam 2008) Let S be the set of all 2×2 matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices $M \in S$ for which there is some integer $k > 1$ such that M^k is also in S.

More problems.

4. Define $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 1$. Prove that

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n$$

for all $n \geq 1$.

Follow-up question: How would you figure out the version of this formula for a different relation, say, $f_{n+1} = 12f_n + 34f_{n-1}$?

5. (Putnam 2006) Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are vertices of a hypercube in $\mathbb{R}^n$.) Given a vector subspace V of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subset \mathbb{R}^n$ of dimension k, of $Z(V)$.

6. Do there exist square matrices A and B such that $AB - BA = I$?

7. (Putnam 2009) Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \dots, \cos(n^2)$. (For example,

$$d_3 = \det \begin{pmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{pmatrix}.$$

The argument of cos is always in radians, not degrees.)

Evaluate $\lim_{n\to\infty} d_n$.

Follow-up question: What other functions could we put in place of cos (and still make this problem work)?

Even more problems (for the next time, unless we magically have too much time on our hands):

8. Let n and k be positive integers. Say that a permutation σ of $\{1, 2, ..., n\}$ is k-limited if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of all k-limited permutations of $\{1, 2, \dots, n\}$ is odd if and only if $n \equiv 0$ or $1 \mod 2k + 1$.

(There is probably a way to solve this problem directly... but can you figure out what it has to do with linear algebra?)

9. Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m+n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

10. Let A be a 5×10 matrix with real entries, and let A' denote its transpose (so A' is a 10×5 matrix, and the ijth entry of A' is the jith entry of A). Suppose every 5×1 matrix with real entries (i.e. column vector in 5 dimensions) can be written in the form Au where u is a 10×1 matrix with real entries. Prove that every 5×1 matrix with real entries can be written in the form $AA'v$ where v is a 5×1 matrix with real entries.

11. Let n be a positive integer, let A, B be square symmetric $n \times n$ matrices with real entries (so if a_{ij} are the entries of A, the a_{ij} are real numbers and $a_{ij} = a_{ji}$). Suppose there are $n \times n$ matrices X, Y (with complex entries) such that

$$\det(AX + BY) \neq 0.$$

Prove that

$$\det(A^2 + B^2) \neq 0$$

(det indicates the determinant).

12. (I really hate this one) Let I denote the 2×2 identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and let

$$M = \begin{pmatrix} I & A \\ B & C \end{pmatrix}, N = \begin{pmatrix} I & B \\ A & C \end{pmatrix},$$

where A, B, C are arbitrary 2×2 matrices with entries in $\mathbb{R}$, the real numbers. Thus M and N are 4×4 matrices with entries in $\mathbb{R}$. Is it true that M is invertible (i.e. there is a 4×4 matrix X such that $MX = XM =$ the identity matrix) implies N is invertible? Justify your answer.

Here is my cheatsheet: a few useful facts and concepts from linear algebra (and I am sure I am forgetting some). Sound familiar?

- Determinants. They...
  - determine whether (a) a matrix is invertible, (b) its rows are linearly independent, (c) its columns are linearly independent;
  - can be expanded recursively by row or by column;
  - can be expressed as a sum over permutations;
  - change in a controllable way under row and column manipulations;
  - multiply when matrices get multiplied.
- Matrices:
  - Rank can be determined by columns and by rows.
  - Eigenvalues, eigevectors, characteristic polynomial, the Cayley-Hamilton Theorem.
- Vector spaces:
  - Basis, dimension, linear (in)dependence, and span.
  - Change of basis.