Line Bundles on Bun_G

Kevin Dao

August 20, 2025

WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

*Mistakes and typos are to be expected. Use with caution.

Goals of this Talk

We already had 2-3 talks about this topic already.

- The goal of this talk is to sort of clean up details and clarify some of the proofs.
- Work out more examples regarding the affine Grassmannian (which I promised in previous discussion).
- Talk some more about line bundles on Bun_G. In particular, the determinant line bundle and the Pfaffian line bundle.
- Describe in more detail the Picard group of Bun_G.
- Compute examples for various cases e.g. $G = \mathrm{SL}_r$ and $G = \mathrm{SO}_r$.
- For the next talks, perhaps someone should give a genuine introduction to (a) affine lie algebras or (b) discuss space of vacua or (c) talk about Harder-Narasimhan stratification.

Recollections on Affine Grassmannians

Recall that $\mathrm{Gr}_G(-)$ is an ind-proper of finite type ind-scheme when G is a connected reductive group. It is reduced iff $\mathrm{Hom}(G, \mathbb{G}_m) = 1$.

Example

Let's just look at $\mathrm{Gr}_G(\mathbb{C})$ for some examples.

- If $G = \mathbb{G}_a$, then

$$\mathrm{Gr}_G(\mathbb{C}) = (\mathbb{C}((t)))/(\mathbb{C}[[t]]) = \left\{\cdots + a_{-2}t^{-2} + a_{-1}t^{-1} + a_0 \mid a_i \in \mathbb{C}\right\} = \mathbb{A}^{\infty}.$$

  Notice that this is not ind-projective. Here, G is not reductive. It is ind-reduced since $\mathrm{Hom}(G, \mathbb{G}_m) = 1$.

- If $G = \mathrm{GL}_1$, then

$$\mathrm{Gr}_G(\mathbb{C}) = \mathbb{C}((t))^{\times}/\mathbb{C}[[t]]^{\times} = \{t^n \mid n \in \mathbb{Z}\} = \mathbb{Z}.$$

  The equality follows by factoring out the lowest power of t that appears e.g.

$$\frac{1}{t^n} + t + t^2 + t^3 + \cdots = t^{-n}(1 + t^{n+1} + t^{n+2} + t^{n+3} + \dots).$$

- (Exercise) Show that $\mathrm{Gr}_G(D) = \mathbb{C}^{\infty} \times \mathbb{Z}$ where D is the ring of dual numbers over $\mathbb{C}$ and $G + \mathbb{G}_1$. **This show that $\mathrm{Gr}_{\mathrm{GL}_1}$ cannot be ind-reduced already.**

Affine Grassmannians (other interesting facts)

Example

Let $G := \mathrm{GL}_n$. What does $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ look like?
It has a bunch of connected components each of which are isomorphic to $\mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C})$.

Theorem (Basic Properties of Affine Grassmannians)

- *If* $G = G_1 \times G_2$, *then* $(\mathrm{Gr}_G)_{red} = (\mathrm{Gr}_{G_1})_{red} \times (\mathrm{Gr}_{G_2})_{red}$.
- *The inclusion* $\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$ *induces an isomorphism between* $\mathrm{Gr}_{\mathrm{SL}_n}$ *and the reduced* 0*-th connected component of* $\mathrm{Gr}_{\mathrm{GL}_n}$.
- $\mathrm{Gr}_{PGL_n}(\mathbb{C}) \cong \mathrm{Gr}_{GL_n}(\mathbb{C})/\mathbb{Z}$.
- $\pi_0(\mathrm{Gr}_G(\mathbb{C})) = \pi_1(G(\mathbb{C}))$.

We can already see these examples in some of the cases above.

Here's a proof for your convenience.

Proof.

- Let $G = G_1 \times G_2$. At the level of $\mathbb{C}$ points, $\mathrm{Gr}_G(\mathbb{C}) = \mathrm{Gr}_{G_1}(\mathbb{C}) \times \mathrm{Gr}_{G_2}(\mathbb{C})$. So the statement follows for the reduced ind-schemes by the following fact.
  Fact. Over $\mathbb{C}$, if X is ind-finite type and ind-reduced, then its closed points are its $\mathbb{C}$-points and $\mathrm{Hom}(X, Y) = \varprojlim_i \varinjlim_i \mathrm{Hom}(X_i, Y_j) = \varprojlim_i \varinjlim_j \mathrm{Hom}(X_i(\mathbb{C}), Y_j(\mathbb{C}))$.
- Bijectivity at the level of $\mathbb{C}$-points is a linear algebra problem. Then apply the above fact.
- Exercise in linear algebra.
- We've seen this already.

□

Line Bundles on Gr_G

Theorem

Assume G is a simply connected simple group. There is a homomorphism $\mathrm{Pic}(\mathrm{Gr}_G) \to \mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}})$ which is an isomorphism.

So this is a theorem of Kumar-Narasimhan-Ramananthan and Sorger only cites it. I at least want to give examples and explanations since we've talked about it in the past. A proof sketch was written down in Jeremy's talk as well.

First, let me describe this homomorphism more directly without too much of language of affine Lie algebras.

Pick the map $\mathrm{SL}_2 \to L\,\mathrm{SL}_2$ which sends $\mathrm{SL}_2(R) \to \mathrm{SL}_2(R((z)))$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix}.$$

Then pick $L\,\mathrm{SL}_2 \to L\widetilde{G}$. Then the borel subgroup of $\mathrm{SL}_2(R)$ goes to $L^+\widetilde{G}$ by construction (this explains the swap between d, a and c, z).

Quotient by Borel and $L^+\widetilde{G}$ to get a map $\bar{\varphi} : \mathbb{P}^1_{\mathbb{C}} \to \mathrm{Gr}_{\widetilde{G}}$.

Then the map in theorem is the pullback map!

Let me state the Cartan Decomposition of the Affine Grassmannian.

Theorem

There is a bijection between

$$X_*(G,T)_+ \leftrightarrow \{G(\mathbb{C}[[t]]) - orbits\ of\ \ \mathrm{Gr}_G(\mathbb{C})\}$$

sending λ *to* $G(\mathbb{C}[[t]])t^\lambda$. *The* $G(\mathbb{C}[[t]])t^\lambda$ *called Schubert cells.*

My intuition is then this. First, it is known that if G is a simple and simply connected group, then $\mathrm{Pic}(G) = 0$. Then using the Cartan decomposoition, there is a decomposition of Gr_G into $G(\mathbb{C}[[t]])$. One can reason that for most groups e.g. $G = \mathrm{SL}_2$, the Picard group of the Schubert cells are trivial. So one can ask about the Picard group of the whole by considering how to patch. So, it seems somewhat reasonable to expect $\mathbb{Z}$ as the Picard.
(I'm not sure if this can be made more precise...I haven't written out details).

Fix your favoriate algebraic curve X. Recall from the uniformization theorem that $\mathrm{Gr}_G \to \mathrm{Bun}_G(X)$ is an étale locally trivial $L_X G$-bundle where $L_X G$ is given by $U \mapsto G(O(X_U)^\times)$. The statement for the uniformization theorem works for semisimple groups[1] at least.

Theorem

From the uniformization theorem, $\mathrm{Pic}(\mathrm{Bun}_G(X))$ *is isomorphic to* $\mathrm{Pic}_{L_X G}(\mathrm{Gr}_G)$ *the group of* $L_X G$*-linearized line bundles on* Gr_G*. The map is induced by* π^* *where* $\pi : \mathrm{Gr}_G \to \mathrm{Bun}_G(X)$*.*

[1](?do recent papers deal with the case of general reductive groups?

More General Groups

Example (What about $G = \mathrm{GL}_n$?)

A lot of the material in Sorger's notes assume G is simply connected $\pm$ (almost) simple.
If $G = \mathbb{G}_m$, then issues already arise.
If $G := \mathbb{G}_m$, then $\mathrm{Bun}_G(X) = \mathrm{Pic}^0(X) \times \mathbb{Z} \times B\mathbb{G}_m$. Line bundles on $\mathrm{Pic}^0(X)$ can be quite complicated e.g. if X is an elliptic curve then $\mathrm{Pic}^0(X) \cong X$.

Example

What if $G = \mathrm{GL}_n$ for $n > 1$? This is the case of vector bundles on a curve.
Again, one would expect that the answer for Pic to have a dicrete piece and a nodiscrete piece. A natural stepping stone would then be to compute the Picard group of the moduli space of rank r semistable vector bundles of fixed determinant L: $M_r^L(X)$. For elliptic curves X, $M_r^0(X) \cong X$ and so this is not too bad.

Theorem (Drézet-Narasimhan, Hoffmann)

$\mathrm{Pic}(M_r^L(X)) = \mathbb{Z}$. *Hoffmann generalized this to algebraically closed fields of any characteristic.*

More General Groups II

Theorem

Assume $g(X) > 0$. Let G be a simple complex group.

1. *There is a canonical isomorphism $\mathrm{Tor}(\mathrm{Pic}(\mathrm{Bun}_G^0(X))) \xrightarrow{\sim} H^1(\widehat{X, \pi_1}(G))$ where the hat denotes Pontrjagin duality: $\widehat{A} = \mathrm{Hom}(A, \mathbb{G}_m)$.*
2. *The quotient $\mathrm{Pic}(\mathrm{Bun}_G^0(X))/\mathrm{Tor}$ is an infinite cyclic group whose positive generator $\mathcal{L}$ is*

$$f_\pi^* \mathcal{L} \cong O_{\mathrm{Bun}_{\widetilde{G}}(X)}(\ell_b)$$

where $\pi : \widetilde{G} \to G$ is the universal cover and $f_\pi : \mathrm{Bun}_{\widetilde{G}}(X) \to \mathrm{Bun}_G(X)$ is given by extension of structure groups. The number ℓ_b is called the basic index whose values I omit (it depends on G e.g. type E_6 has $\ell_b = 3$).

It is not clear to me why $g(X) = 0$ is excluded in this story in the lecture notes. The original paper makes no assumption of $g(X) > 0$ either.

More General Groups III

Proof of Theorem.

(1) will be proved. (2) is omitted. **(2) is proven by looking at the map**$\mathrm{Pic}(\mathrm{Bun}_G^0(X)) \to \mathrm{Pic}(\mathrm{Gr}_{\widetilde{G}})$**.**

From previous discussion, the forgetful map $f : \mathrm{Pic}_{L_X\widetilde{G}}(\mathrm{Gr}_{\widetilde{G}}) \to \mathrm{Pic}(\mathrm{Gr}_{\widetilde{G}}) \cong \mathbb{Z}$ has kernel given by the character group of $L_X G$. Look at the exact sequence

$$1 \to L_X\widetilde{G}/\pi_1(G) \to L_X G \to H^1(X^*, \pi_1(G)) \to 1$$

The characters of $L_X\widetilde{G}$ are trivial and so characters of $L_X G$ are given by characters of $H^1(X^*, \pi_1(G))$. But the restriction map $H^1(X, \pi_1(G)) \to H^1(X^*, \pi_1(G))$ is bijective so we win.

Technical gap: Why does the former not have larger free part that just maps to zero?

Explanation: As written this uses part (2). Why the proof passes through f uses the next slide. □

$$
\begin{array}{ccc}
\mathrm{Pic}(\mathrm{Bun}^0_G(X)) & \xrightarrow{f_\pi^*} & \mathrm{Pic}(\mathrm{Bun}_{\widetilde{G}}(X)) \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
\mathrm{Pic}_{L_X G}(\mathrm{Gr}^0_G) & \longrightarrow & \mathrm{Pic}_{L_X \widetilde{G}}(\mathrm{Gr}_{\widetilde{G}}) \\
\text{kernel is } \mathrm{Hom}(L_X G, \mathbb{G}_m) \Big\downarrow & & \Big\downarrow \\
\mathrm{Pic}(\mathrm{Gr}_G) & & \mathrm{Pic}(\mathrm{Gr}_G) \\
& \searrow \cong & \Big\downarrow \cong \\
& & \mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}})
\end{array}
$$

Examples with SL_r I

I want to just work out the example of $G = \mathrm{SL}_r$ with X a projective smooth complex curve. Since SL_r is simply connected, I know that $\mathrm{Pic}(\mathrm{Bun}_G(X)) = \mathbb{Z}$ and its generator arises from the generator of $\mathrm{Pic}(\mathrm{Gr}_G)$.

Theorem

The determinant bundle D on $\mathrm{Bun}_{\mathrm{SL}_r}(X)$ is $O_{\mathrm{Bun}_G(X)}(1)$.

Examples with SL_r II

Here's what the determinant bundle is.

First off, to give a line bundle L on an algebraic stack $\mathcal{X}$, one must prescribe to each (smooth) morphism $S \to \mathcal{X}$ (from a scheme), a line bundle $\mathcal{L}_S$ such that for all composites $S' \xrightarrow{f} S \to \mathcal{X}$, one gives an isomorphism $f^{-1}\mathcal{L}_S \xrightarrow{\sim} \mathcal{L}_{S'}$.

Every map $S \to \mathrm{Bun}_G(X)$, by the Yoneda Lemma is givenby some family $\mathcal{E}_S$ of vector bundles over X parameterized by S. We would like to extract a line bundle from $\mathcal{E}_S$. We *could potentially* just take determinants e.g. to each $S \to \mathrm{Bun}_G(X)$ we just associate $\det \mathcal{E}_S$ which is a line bundle on $X_S := X \times S$, but this cannot be right! We want a line bundle on S. Not on X_S. So clearly this process isn't the right way to do so!

Here's the correct process.

First, resolve $\mathcal{E}_S$ by S-flat coherent sheaves $0 \to P_1 \to P_0 \to F \to 0$. By Serre's Theorem A it is possible to make $p_* P_0$. So pushforward along $p : X_S \to S$ to get an exact sequence $0 \to R^1 p_* P_1 \to R^1 p_* P_0 \to 0$. This gives a length one perfect complex representing $Rp_*\mathcal{E}_S$ in $D^b_c(S)$.

We then set $\mathcal{D}_{\mathcal{E}_S} := \det(Rp_*\mathcal{E}_S)^{-1}$ where the determinant of a two-term complex $0 \to K^0 \to K^1 \to 0$ of locally free sheaves is $\wedge^{top} K^0 \otimes (\wedge^{top} K^1)^{-1}$. '

Theorem

$\mathcal{D}_{\mathcal{E}_S}$ does not depend up to canonical isomorphism on the choice of perfect complex representing $Rp_*\mathcal{E}_S$. Furthermore, $\mathcal{D}_{\mathcal{E}_S} \in \mathrm{Pic}(S)$.

Examples with SL_r III

Definition

The determinant line bundle $\mathcal{D}$ on $\mathrm{Bun}_G(X)$ associates to each smooth morphism $\mathcal{E}_S : S \to \mathrm{Bun}_G(X)$ the line bundle $\mathcal{D}_{\mathcal{E}_S} \in \mathrm{Pic}(S)$.
The fibres of $\mathcal{D}_{\mathcal{E}_S}$ over s are given by $(\wedge H^0(X, F_s))^{-1} \otimes \wedge H^1(X, F_s)$.

It is perhaps worth noting thatone can always twist the families $\mathcal{E}_S$ by bundles coming from from X. Doing so gives twists of the determinant line bundle. It is a theorem of Le Potier that this depends only on the K-class of the bundles you twist by.
As part of training, let's work out what happens when $G = \mathrm{SL}_r$.

Lemma

Let F be a vector bundle on X_S such that $\wedge F$ is the pullback of some line bundle along $X_S \to X$. Then

$$\mathcal{D}_{F \otimes q^* u} = \mathcal{D}_F^{rank(u)}$$

where u is a vector bundle on X.

Proof.

Via some reductions one can assume u is given by $O_X(-p)$. Consider the modification sequence $0 \to O_X(-p) \to O_X \to O_p \to 0$. Note that $\mathcal{D}_{F \otimes q^* O_p}$ is the trivial line bundle by our assumptionon F. So, $\mathcal{D}_{F \otimes q^* u} \cong \mathcal{D}_F^{\otimes rank(u)}$ since $rank(u) = 1$. □

Examples with SL_r IV

Let's use what we know about determinant line bundles to determine that $\mathrm{Pic}(\mathrm{Bun}_G(X)) \cong \mathbb{Z}\mathcal{D}$ with $\mathcal{D}$ the ample generator. Again, $G = \mathrm{SL}_r$ is simply connected so our previous results apply.

Examples with SL_r IV

There's a diagram

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbb{C}} & \xrightarrow{\varphi} & \mathrm{Gr}_{\mathrm{SL}_r} \\ & & \downarrow \pi \\ & & \mathrm{Bun}_{\mathrm{SL}_r}(X) \end{array}$$

The composite $\pi \circ \varphi : \mathbb{P}^1_{\mathbb{C}} \to \mathrm{Bun}_{\mathrm{SL}_r}(X)$ gives a family $E := \mathcal{E}_{\mathbb{P}^1_{\mathbb{C}}}$ of SL_r-bundles on X parameterized by $\mathbb{P}^1_{\mathbb{C}}$.
Let's just pick $r = 2$ for ease (one can justify that this is sufficient to prove the statement).
For each point $[a : c] \in \mathbb{P}^1_{\mathbb{C}}$, we get some SL_2-bundle on X which we denote by $E_{[a:c]}$.
Using the map to $\mathrm{Gr}_{\mathrm{SL}_2}$, we get an SL_2-bundle on the formal disk given by the lattice

$$W = \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} (\mathbb{C}[[z]] \oplus \mathbb{C}[[z]]) \hookrightarrow \mathbb{C}((z)) \oplus \mathbb{C}((z)).$$

Consider the lattice $V = z^{-1}\mathbb{C}[[z]] \oplus \mathbb{C}[[z]]$ which corresponds to the bundle $F = O_X(p) \oplus O_X$. There's the inclusion $W \subseteq V$ which corresponds to the modification $E_{[a:c]} = \ker(F \to O_p)$ (use the fact that the lattice W has determinant 1). The map $F \to O_p$ sends $(z^{-1}f, g) \to af(p) - cg(p)$. Then that means the determinant of $E_{[a:c]}$ is going to give me $\mathcal{D}_E = O_{\mathbb{P}^1_{\mathbb{C}}}(1)$ (remember the dual).

Pfaffian Line Bundle

We must explicitly work in characteristic $\neq 2$. Otherwise the material here is not quite right.
A vector bundle E over X_S has a **nondegenerate quadratic form valued in** ω_X if there is an isomorphism $\sigma : E \to E^\vee$ s.t. $\sigma = \sigma^\vee$ where $E^\vee = Hom_{O_{X_S}}(E, q^*\omega_X)$ where $q : X_S \to X$ is the projection.

Lemma (Lemmas on Representative for Cohomology and Skew Complexes)

$0 \to K^0 \xrightarrow{\alpha} K^1 \to 0$ *is skew if* α *is skew-symmetric.*

Definition

Let F be a vector bundle on X_S equipped with a nondegenerate quadratic form σ valued in ω_X.
Glue the determinant of skew complexes together. This defines a **canonical square root** of $\mathcal{D}_F$ called the **Pfaffian bundle** which is denoted by $\mathcal{P}_{(F,\sigma)}$.

Pfaffian Line Bundle on SO_r

Let ≥ 3 and F be the universal bundle on $\mathrm{Bun}_{\mathrm{SO}_r}(X) \times X$. Take a line bundle κ s.t. $\kappa \otimes \kappa \cong \omega_X$, then $F_\kappa = F \otimes q^*\kappa$ has a nondegenerate quadratic form with values in ω (even if κ is trivial). Then the previous theorem implies there is a Pfaffian line bundle associated to F_κ denoted $\mathcal{P}_\kappa$ for short.

When is the Pfaffian Line Bundle a Generator?

Fix $\widetilde{G}$ a simply connected simple group. We know $\mathrm{Pic}(\mathrm{Bun}_{\widetilde{G}}(X)) \cong \mathbb{Z}$ is infinite cyclic. We ask what the generator is in the cases of type B, D, G_2. The main fact is the following:

Theorem

For types B, D, G_2 (and hence the case where $G = \mathrm{SO}_r$ and $r \geq 3$), we have $\mathrm{Pic}(\mathrm{Bun}_{\widetilde{G}}(X)) \cong \mathbb{Z}\mathcal{P}$.

We first introduce the **Dynkin index**. Take $\widetilde{G}$ and consider a representation $\rho : \widetilde{G} \to \mathrm{SL}_r$. Then we get the extension of scalars map $f_\rho : \mathrm{Bun}_{\widetilde{G}}(X) \to \mathrm{Bun}_{\mathrm{SL}_r}(X)$. Then the pullback map is $f_\rho^* : \mathrm{Pic}(\mathrm{Bun}_{\mathrm{SL}_r}(X)) \to \mathrm{Pic}(\mathrm{Bun}_{\widetilde{G}}(X))$ and both groups are canonically isomorphic to $\mathbb{Z}$. The map is injective. **Ehh?? Is this clear? Claim:** the index of this map is the Dynkin index d_ρ. We realize d_ρ by looking at the level of the affine Grassmannian $f_\rho^* O_{\mathrm{Gr}_{\mathrm{SL}_r}}(1) =: O_{\mathrm{Gr}_{\widetilde{G}}}(d_\rho)$.

Since $\widetilde{G}$ is a simply connected simple group, we can "integrate" and study representations of the Lie algebra $\mathfrak{g}$. Set $d_{\mathfrak{g}}$ as the GCD over all d_ρ arising from representations of $\mathfrak{g}$. Let D_ρ denote the pullback of the determinant bundle along $\mathrm{Bun}_{\widetilde{G}}(X) \to \mathrm{Bun}_{\mathrm{SL}_r}(X)$.

When is the Pfaffian Line Bundle a Generator?

Then, the above follows from

Theorem

$[\mathrm{Pic}(\mathrm{Bun}_{\widetilde{G}}(X) : \mathrm{Pic}_{det}(\mathrm{Bun}_{\widetilde{G}}(X)] = d_{\mathfrak{g}}$ where $\mathrm{Pic}_{det}(\mathrm{Bun}_{\widetilde{G}}(X)$ is the subgroup generated by all of the D_ρ ranging over all representations.

Recall that $O_{\mathrm{Gr}_{\widetilde{G}}}(1)$ had an associated Mumford group which was a central extension

$$1 \to \mathbb{G}_m \to \widetilde{L\widetilde{G}} \to L\widetilde{G} \to 1.$$

We also have a similar exact sequence for SL_r. Differentiating, we get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{L\mathfrak{g}} & \longrightarrow & L\mathfrak{g} & \longrightarrow & 0 \\ & & \Big\Vert = & & \downarrow & & \downarrow L\rho & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{L\mathfrak{sl}_r} & \longrightarrow & L\mathfrak{sl}_r & \longrightarrow & 0 \end{array}$$

To determine what canonical central extension we get for $\widetilde{L}\mathfrak{g}$, one computes the cocycle. This comes in the form of a lemma.

In previous talks, we had discussed $O_{\mathrm{Gr}_{\widetilde{G}}}(1)$ and its construction. The precise relationship is spelled out for convenience.

Theorem

On $\mathrm{Gr}_{\widetilde{G}}$, there is a line bundle $O_{\mathrm{Gr}_{\widetilde{G}}}(1)$ which is built from the character

$$\chi : \widehat{L^+\widetilde{G}} \to \mathbb{G}_m \times L^+\widetilde{G} \xrightarrow{p_1} \mathbb{G}_m.$$

Here, we had a canonical central extension $0 \to \mathbb{G}_m \to \widehat{L\widetilde{G}} \to L\widetilde{G} \to 1$ (arising from representation theory). The first map above arose from the splitting of this central extension upon restriction to $L^+\widetilde{G}$.
Note

$$\mathrm{Gr}_{\widetilde{G}} = \widehat{L\widetilde{G}}/\widehat{L^+\widetilde{G}} = LG/L^+G.$$

Furthermore, $\mathrm{Pic}(\mathrm{Gr}_{\widetilde{G}}) \cong \mathbb{Z}$.

It is worth noting that Kumar et. al. prove the theorem through an application of GAGA in “Infinite Grassmannians and moduli spaces of G-bundles”.

Lemma

The cocycle for V is given by

$$\left(\frac{1}{2}\sum_{\lambda} n_{\lambda}\lambda(H_{\theta})^{2}\right)\psi_{\mathfrak{g}}$$

The quantity in front from $\psi_{\mathfrak{g}}$ is $d_{\mathfrak{g}}$.

If we return to the theorem regarding types B, D, G_2, one then only needs to observe (a) the Pfaffian makes sense in these cases and (b) the Dynkin index in all these cases equals 2.

Note: I haven't gone of out of my way to compute the Dynkin indices myself. Sorger gives this in a table. Exercise.

Examples with SO_r where $r \geq 3$

Let $G := \mathrm{SO}_r$. There is an exact sequence

$$0 \to H^1(\widehat{X, \pi_1}(G)) = (\mathbb{Z}/2\mathbb{Z})^{2g} = J_2 \to \mathrm{Pic}(\mathrm{Bun}_G^0(X)) \to \mathbb{Z} \to 0.$$

The torsionfree part $\mathbb{Z}$ is generated by the Pfaffian $\mathcal{P}_\kappa = \mathcal{P} \otimes q^*\kappa$ for any theta characteristic κ (i.e. $\kappa^{\otimes 2} \cong \omega_X$ and here we use the Pfaffian w.r.t. the universal bundle). Now let $\theta(X) \subseteq \mathrm{Pic}(X)$ denote the subgroup generated by theta characteristics. Then the Pfaffian $\mathcal{P} : \theta(X) \to \mathrm{Pic}(\mathrm{Bun}_G^0(X))$ is an isomorphism. Here, J_2 is the kernel of $[2] : Jac(X) \to Jac(X)$.

Question: Why is the torsionfree part generated by the Pfaffian? See previous slide.