Local operators, bilayer Turing reducibility, and enumeration Weihrauch degrees

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Introduction

Toposes and local operators

A **topos** $\mathcal{E}$ is a category that

- is finitely complete (has finite products and equalizers),
- is cartesian closed (for any objects $X, Y \in \mathcal{E}$ we can form the exponential object Y^X with an evaluation morphism $X \times Y^X \to Y$)
- contains an object Ω such that for any $X \in \mathcal{E}$, morphisms $X \to \Omega$ correspond naturally to subobjects of X.

From these we can form "power objects" in the form of Ω^X, and evaluation $X \times \Omega^X \to \Omega$ acts like a membership predicate.

One can also define truth functional operations $\neg, \wedge, \vee, \Rightarrow$ as maps $\Omega^i \to \Omega$ (with i the arity of the connective), and quantifiers $\forall_X, \exists_X$ as maps $\Omega^X \to \Omega$.

Toposes and local operators

A **local operator** (or **Lawvere-Tierney topology**) on a topos $\mathcal{E}$ is a morphism $j : \Omega \to \Omega$ such that

- $jj = j$
- $j \circ \wedge = \wedge \circ (j \times j)$
- $j\top = \top$

Local operators strengthen the higher order theory modeled by $\mathcal{E}$, and induce subtoposes of "j-sheaves". Some examples of local operators:

- $\mathbf{1}_\Omega$ (induces the total subtopos)
- $\top_\Omega$ (the "inconsistency operator", induces the trivial subtopos.)
- $\neg\neg : \Omega \to \Omega$ (induces a subtopos with a Boolean internal logic)
- For a morphism $p : 1 \to \Omega$, $p \Rightarrow -$ and $p \vee -$ are local operators.

Toposes and local operators

The important idea: toposes model a higher order intuitionistic logic/typed set theory. Local operators fine tune this theory. For example,

- The category of sets in your average Boolean-valued model of set theory is essentially the $\neg\neg$-sheaves on $\mathbf{Set}^H$, for some complete Heyting algebra H.
- Classifying toposes are built around the syntax of a first order theory, and local operators correspond to the extensions of said theory.
- Presheaf toposes carry loose information about a topological space (X, τ), but the local operators give categories of sheaves (imposing coherence conditions).

Realizability

Realizability semantics are a semantics for non-classical logic that interprets sentences computationally. The original definition of realizability for first order arithmetic is due to [Kleene, 1952], and has since been vastly generalized.

The rough idea: to each predicate ϕ associate a function $\phi : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, interpreting

- $\phi(n) \wedge \psi(n)$ as $\{\langle m, k\rangle \mid m \in \phi(n) \wedge k \in \psi(n)\}$,
- $\phi(n) \Rightarrow \psi(n)$ as $\{e \mid (\forall m \in \phi(n))(\varphi_e(m) \in \psi(n))\}$,
- etc.

To talk about the common generalizations, we need to generalize our notion of computation.

Schönfinkel algebras

A **Schönfinkel algebra** (or **partial combinatory algebra**) is a set A equipped with

- a partial binary operation $A \times A \to A$ (usually called "application".
- an element $\mathtt{k} \in A$ such that for all $a \in A$ we have $\mathtt{k}a\downarrow$ and $\mathtt{k}ab = a$ for any $b \in A$,
- and an element $\mathtt{s} \in A$ with $\mathtt{s}ab\downarrow$ and $\mathtt{s}abc \simeq (ac)bc$ for all $a, b, c \in A$

Schönfinkel algebras

Named for Moses Schönfinkel, who proposed a similar formalism in [Schönfinkel, 1924] that was equivalent to the lambda calculus. The present notion is due to [Feferman, 1975].

A Schönfinkel algebra gives a notion of computation; in any non-trivial such algebra, we can encode the natural numbers and partial recursive functions—and, potentially, more.

Schönfinkel algebras

Key examples:

- The first Kleene algebra $\mathcal{K}_1$. The underlying set is $\mathbb{N}$, and application is defined by $n \cdot m = \varphi_n(m)$.
- The second Kleene algebra $\mathcal{K}_2$. The underlying set is $\mathbb{N}^{\mathbb{N}}$; the application operation $f \cdot g$ is $n \mapsto k - 1$ where k is the first non-zero value of $f(n^\frown g \upharpoonright i)$ if this is defined for all n, and undefined otherwise.
- The Scott algebra $\mathbb{S}$. The underlying set is $\mathcal{P}(\mathbb{N})$, and for $A, B \in \mathbb{S}$, $AB = \{x \mid \langle x, n\rangle \in A \wedge D_n \subseteq B\}$

Schönfinkel algebras

Given a Schönfinkel algebra A, one can build a topos structured by A, called a **realizability topos**. The construction is due to [Hyland et al., 1980].

An object in a realizability topos is a pair $(X, \approx)$ where X is a set, and $\approx$ is a function $X \times X \to \mathcal{P}(A)$ such that

- there is some $a \in A$ such that for all $x, y \in X$ and $b \in x \approx y$, $a \cdot b \downarrow \in y \approx x$,
- there is some $a \in A$ such that for any $x, y, z \in X$, $b_0 \in x \approx y$, and $b_1 \in y \approx z$, $a \cdot \langle b_0, b_1 \rangle \downarrow \in x \approx z$.

Sometimes we are interested in restricting the a's above to an elementary subalgebra $A_\# \subset A$. In which case the pair $(A, A_\#)$ is a *relative* Schönfinkel algebra, and the resulting topos is a relative realizability topos.

Schönfinkel algebras

A couple of realizability toposes:

- $\mathcal{K}_1$ gives the **effective topos** (**Eff**).
- $\mathbb{S}$ gives the **Scott realizability topos** ($\mathrm{RT}(\mathbb{S})$).

A couple of relative realizability toposes:

- $\mathcal{K}_2$ with its subalgebra of computable elements gives the **Kleene-Vesley topos**.
- $\mathbb{S}$ with its subalgebra of c.e. elements gives the **relative Scott realizability topos**.

Schönfinkel algebras

It's tempting to view local operators on a realizability topos as somehow strengthening the underlying notion of computation, and the lattice of local operators as a degree structure.

- There's a good notion of adjoining an oracle to a Schönfinkel algebra, and each oracle added to $\mathcal{K}_1$ gives a local operator on **Eff**.
- In fact, the Turing degrees embed (as a poset) into the local operators on **Eff**.
- But not every local operator arises this way. Some extended notions of oracle computation (e.g. multifunction oracles) give more local operators.

Is there a notion of oracle computation whose degrees structure is isomorphic to the lattice of local operators on Eff?

Yes. Bilayer Turing degrees.

Bilayer functions

Recently, [Kihara, 2021] has been able to formalize this notion of local operators as a degree structure in more traditionally computability theoretic language, as so-called **bilayer Turing reducibility**.

(Everything in this section is entirely due to Kihara.)

Bilayer functions

A **bilayer function** f is a partial multifunction $f :\subseteq \mathbb{N} \times \Lambda \rightrightarrows \mathbb{N}$, where Λ is some set. We write $\langle n, c \rangle$ in the domain as $(n|c)$, calling n the *public input* and c the *private input*.

Bilayer functions

Why are these the right sorts of things to look at?

- A local operator in **Eff** is a function $j : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ (with some properties to be mentioned later)
- Every such function corresponds to a unique $R_j \subseteq \mathbb{N} \times \mathcal{P}(\mathbb{N})$.
- We can then define the bilayer function $\tilde{j}$ with domain R_j, and with $j(n|X) = X$.
- Moreover, every bilayer function "essentially" looks like this.

So, up to adding "extra copies" of elements in the domain, bilayer functions just repackage maps $\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$; the trick is finding the right notion of reduction.

Bilayer Turing reduction

We can describe bilayer Turing reducibility game theoretically. Given bilayer functions f, g, let us consider the three player imperfect information game $\mathfrak{G}(f, g)$ between Merlin, Arthur, and Nimue:

- At round 0, Merlin plays $(x_0|c)$ in the domain of f.
- At round n Arthur plays either $\langle 0, u_n\rangle$ with u_n in the public domain of g, or plays $\langle 1, u_n\rangle$, declaring the end of the game with u_n.
- At round n Nimue plays y_n such that $g(u_n|y_n) \neq \varnothing$.
- At round $n+1$, Merlin plays some $x_{n+1} \in g(u_n|y_n)$.

Arthur and Nimue win if Arthur plays $\langle 1, u_n\rangle$ with $u_n \in f(x_0|c)$.

Bilayer Turing reduction

We say that f is bilayer Turing reducible to g, denoted $f \leq_{LT} g$ when Arthur and Nimue have respective strategies τ and η such that

- τ is computable and can only see the x_i (though Merlin and Nimue's strategies may see all previous moves);
- τ, η are jointly winning; when Arthur and Nimue play $\mathfrak{G}(f, g)$ using these strategies, they win regardless of Merlin's strategies.

If Arthur & Nimue have a winning strategy using exactly one oracle query, we write $f \leq^1_{LT} g$.

Note that if f, g are single-valued, total, and don't depend on the private inputs, this will reduce to ordinary Turing reducibility between functions.

Turns out, the poset of bilayer Turing degrees is equivalent to the lattice of local operators in **Eff**.

Examples of Bilayer functions

- Any function $f : \mathbb{N} \to \mathbb{N}$ can be construed as a bilayer function $\bar{f} :\subseteq \mathbb{N} \times \{*\} \rightrightarrows \mathbb{N}$ where $\bar{f}(n|*) = \{f(n)\}$.
- $\texttt{Cofinite}$, where $\mathrm{dom}(\texttt{Cof}) = \{*\} \times \mathbb{N}$ and $\texttt{Cof}(*|k) = \{n \mid n > k\}$.
- $C^{\alpha}_{\texttt{cof}}$ with domain those $(e|*)$ such that $\{n \mid \varphi^{\alpha}_{e}(n)\downarrow\}$ is finite, and with $C^{\alpha}_{\texttt{cof}}(e|*) = \{n \mid \varphi^{\alpha}_{e}(n)\uparrow\}$
- $\texttt{LLPO}_{m/k}$, with domain $\{e \mid |\{n \mid n \leq k \wedge \varphi_e(n)\downarrow\}| \leq m\}$ and $\texttt{LLPO}_{m/k}(e) = \{n \mid \varphi_e(n)\uparrow\}$.

$\texttt{Cofinite}$ is a good example of the ridiculous strength of some bilayer functions. Everything below it on this list reduces to it. More than that, for a function $f : \mathbb{N} \to \mathbb{N}$, $f \leq_{LT} \texttt{Cofinite}$ iff f is hyperarithmetic [van Oosten, 2014].

Local operators from bilayer functions

For $X, Y \in \mathcal{P}(\mathbb{N})$, let $X \Rightarrow Y$ be the set of all e such that for all $n \in X$, $e \cdot n \downarrow \in Y$.

In **Eff**, a local operator is represented by a map $j : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that

- there is an $e \in \mathbb{N}$ such that for any $X \in \mathcal{P}(\mathbb{N})$, $e \in X \Rightarrow j(X)$,
- There's an $e \in \mathbb{N}$ such that for any $X \in \mathcal{P}(\mathbb{N})$, $e \in j(j(X)) \Rightarrow j(X)$
- There are e, e' such that for all $X, Y \in \mathcal{P}(\mathbb{N})$, $e \in j(X \times Y) \Rightarrow j(X) \times j(Y)$, and $e' \in j(X) \times j(Y) \Rightarrow j(X \times Y)$.

Local operators from bilayer functions

Given a bilayer function f, define $f^{\rotatebox{180}{$\partial$}}$ to be the bilayer function whose instances are pairs $(\tau|\eta)$ where τ codes an Arthur strategy, η is a Nimue strategy, and any play with these strategies, *where Merlin makes no first move*, ends with Arthur declaring termination of the game.

Define $f^{\rotatebox{180}{$\partial$}}(\tau|\eta)$ to be the set of all values with which Arthur can declare termination. We can think of $f^{\rotatebox{180}{$\partial$}}$ as the "universal machine" with oracle f.

Fact (Kihara, 2.15)

$g \leq_{LT} f$ iff $g \leq^{1}_{LT} f^{\rotatebox{180}{∂}}$.

Local operators from bilayer functions

Given a bilayer function f, define $f^{\rightarrow} : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$ such that $\langle n, e \rangle \in f^{\rightarrow}(X)$ if and only if there is some c such that $e \in f(n|c) \Rightarrow X$.

Think of $f^{\rightarrow}(X)$ as giving you the set of one-query Arthur strategies for getting an element of X using oracle f.

Local operators from bilayer functions

Finally, given a bilayer function f, $f^{\partial\to} : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ represents a local operator in **Eff**. The sketch of a proof is something like this:

- If we can computably produce an element of X, then we can turn that program into an $f^{\partial\to}$-program that doesn't use the oracle (so an element of $X \Rightarrow f^{\partial\to}(X)$);
- To get an element of $f^{\partial\to}(X) \times f^{\partial\to}(Y) \Rightarrow f^{\partial\to}(X \times Y)$, we can parallelize the computations used for each of $f^{\partial\to}(X)$ and $f^{\partial\to}(Y)$. The other direction is easier.
- The way to obtain an element of $f^{\partial\to} f^{\partial\to}(X) \Rightarrow f^{\partial\to}(X)$ is a bit more technical, but the idea is that when the definitions are unpacked, an element of $f^{\partial\to} f^{\partial\to}(X)$ is a strategy for producing strategies to a second game, and we simply run these in sequence.

The remainder gets even more technical, but the operation $f \mapsto f^{\partial\to}$ turns out to be an isomorphism of preorders.

The enumeration setting

(Joint work with Mariya Soskova and Jun Le Goh)

An observation

Nothing in the portion of Kihara's work discussed above depends on any special feature of $\mathcal{K}_1$. We could easily use any other Schönfinkel algebra A.

Moreover, we can use any *relative* Schönfinkel algebra $(A, A_\#)$, requiring that Arthur's strategy be an element of $A_\#$ (and adjusting the definitions of $-^{\ni}$ and $-^{\to}$)

So, in particular, we can examine this notion over the relative Scott algebra $(\mathbb{S}, \mathbb{S}_\#)$, where $\mathbb{S}_\#$ is the subalgebra of c.e. elements. We should be able to make inferences about the local operators of the relative Scott realizability topos, RT$(\mathbb{S}, \mathbb{S}_\#)$, which are not well understood.

$\mathrm{RT}(\mathbb{S}, \mathbb{S}_\#)$

Why study $\mathrm{RT}(\mathbb{S}, \mathbb{S}_\#)$?

- Like **Eff** does for classical computability, $\mathrm{RT}(\mathbb{S}, \mathbb{S}_\#)$ gives a setting where (at least sometimes) more elegant proofs can be found for facts about computation in $(\mathbb{S}, \mathbb{S}_\#)$.
- The underlying notion of computation has connections to enumeration reduction.
- It contains subcategories that have been explored as settings for computable analysis.

RT($\mathbb{S}, \mathbb{S}_\#$)

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- Like **Eff** does for classical computability, RT($\mathbb{S}, \mathbb{S}_\#$) gives a setting where (at least sometimes) more elegant proofs can be found for facts about computation in ($\mathbb{S}, \mathbb{S}_\#$).
- The underlying notion of computation has connections to enumeration reduction.
- It contains subcategories that have been explored as settings for computable analysis.
- It's just cool.

Relative Scott bilayer functions

The notion of a bilayer function is essentially unchanged: a partial multifunction $f :\subseteq \mathbb{S} \times \Lambda \rightrightarrows \mathbb{S}$ for some set Λ.

The reduction game $\mathfrak{G}(f, g)$ looks the same, except now Arthur's strategy must be encoded by an enumeration operator Γ.

The link to local operators, $f^{\circlearrowright}$ must be adjusted so that it's domain is pairs $(\langle A, \tau\rangle | \langle\langle A, c\rangle, \eta\rangle)$ where every play starting from $(A|c)$ following strategies τ, η ends with Arthur declaring termination.

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They desperately need a better name.

The enumeration Weihrauch degrees

Our starting point for understanding "relative Scott bilayer functions" is to understand the simpler, one-query, Nimue-free reductions. Provisionally, we call this *enumeration Weihrauch reduction*.

Concretely, for $f, g :\subseteq \mathbb{S} \rightrightarrows \mathbb{S}$, we say that $f \leq_{eW} g$ if there exist enumeration operators Γ, Δ such that

1. $X \in \mathrm{dom}(f)$ implies $\Gamma(X) \in \mathrm{dom}(g)$,
2. if $Y \in g(\Gamma(X))$, then $\Delta(X \oplus Y) \in f(X)$.

The enumeration Weihrauch degrees

It's still early, but some basic results are easily established:

- There are meets, joins, and no non-trivial countable suprema. These follow from almost literally the same proofs as in the standard Weihrauch setting.
- There is an embedding of the Weihrauch degrees into the *eW* degrees.
  - Take a problem $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ to the problem $\tilde{f} :\subseteq \mathcal{P}(\mathbb{N}) \rightrightarrows \mathcal{P}(\mathbb{N})$, with the instances/solutions of f replaced with their graphs. Replace any functional in a Weihrauch reduction with an enumeration operator that acts like said functional on graphs.
  - Injectivity comes from using an enumeration operator Γ to form a mapping of a finite string σ to the longest τ such that $\text{graph}(\tau) \subseteq \Gamma_{|\sigma|}(\text{graph}(\sigma))$.

The enumeration Weihrauch degrees

The embedding of the previous page is not surjective: there exists a problem $g :\subseteq \mathbb{S} \rightrightarrows \mathbb{S}$ such that $g \not\equiv_{eW} \tilde{f}$ for any $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$.

Proof (M. Soskova, J. Goh).

Let g have only 1-generics in its (non-empty) domain. If $g \leq_{eW} \tilde{f}$, then there's an enumeration operator Γ such that $\Gamma(G) \in \mathrm{dom}(\tilde{f})$ for $G \in \mathrm{dom}(g)$. But since 1-generics are quasi-minimal, and elements in the domain of $\tilde{f}$ have total e-degree, $\mathrm{dom}(\tilde{f})$ contains computable elements.

But if $\tilde{f} \leq_{eW} g$, there must be an enumeration operator e-reducing a 1-generic to a computable set, which is impossible. $\square$

The enumeration Weihrauch degrees

There is also a map of preorders from the *eW*-degrees to the *W*-degrees, via the representation $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{S}$ with $\delta(f) = \mathrm{ran}(f)$. An enumeration Γ induces a functional taking an enumeration of X to an enumeration of ΓX.

It seems unlikely that this is an embedding—not every functional has this form—but we don't quite have a counterexample.

The bigger question: the Weihrauch degrees and the enumeration Weihrauch degrees are certainly not isomorphic by either of the maps above, but might they be isomorphic some other way?

Thanks!

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