Lowness for isomophism

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Contents

Computationally weak sets

- Lowness captures the notion that a particular set is computationally weak.
- Some other examples of notions of being computationally weak are minimal and hyperimmune free degrees.
- We say a set A is low for $\mathcal{P}$ if $\mathcal{P}^A = \mathcal{P}$ that is having A as an oracle doesn't give us anything new.

Notions of Lowness

- A is low (classically) if $A' = 0'$, which can be stated as $\mathcal{P}$ = set of computably approximable objects is the same as the set of objects A can approximate ($\mathcal{P}^A$)
- If $\mathcal{P}$ is the class of *ML* randoms, then A is low for $\mathcal{P}$ if A cannot 'derandomize' any *ML* random set.

Low for isomorphism

A degree d is low for isomorphism if for every pair of computable structures $\mathcal{A}$ and $\mathcal{B}$ we have $\mathcal{A} \cong_d \mathcal{B} \iff \mathcal{A} \cong_{\Delta^0_1} \mathcal{B}$

Equivalent characterizations

We say that a set A is low for paths in Baire (Cantor) space if every Π^0_1 class $\mathcal{P} \subset \omega^\omega$ (2^ω) which has an A computable element has a computable element.

For $A \in 2^\omega$, the following are equivalent

- A is low for isomophism.
- A is low for paths in Baire space.
- A is low for paths in Cantor space.

Comparisons and Examples

- Various forcing notions can be used to produce low for isomorphism degrees
- Some degrees which are not low for isomorphism:
  - degrees which compute a non computable Δ^0_2 set
  - Degrees that compute a separator for computably inseparable c.e. sets.
  - There are hyperimmune free degrees which are not low for isomorphism.

Main Results:

Measure of the class of low for isomophism sets

Let $\mathcal{C}$ be the class of low for isomorphism set. Then $\mu(C) = 0$ where μ is the Lebesgue measure on the unit interval.

Corollary

No *ML* random degree can be low for isomorphism.

Kucera's Theorem

Kucera's Theorem

Let $\mathcal{X}$ be a Π^0_1 class with $\mu(\mathcal{X}) > 0$.
$\forall A$, A is Martin Lof random $\implies \exists\sigma \in 2^{<\omega}, X \in \mathcal{X}$ such that $A = \sigma \frown X$.

Therefore a $ML-$ random computes a member in every Π^0_1 class of positive measure. To prove $\mu(\mathcal{C}) = 0$ below, we construct a Π^0_1 class of positive measure, none of whose elements is low for isomorphism. Therefore no $ML-$ random can be low for isomorphism.

Kolmogrov's $0-1$ law

Definition

Given a sequence of events $\{A_n\}_n$, the tail σ algebra $\mathcal{T}(\{A_n\}_n)$ is defined as $\cap_n \sigma(\{A_m : m > n\})$

Theorem (Kolmogrov's $0-1$ law)

Let $\{A_n\}$ be independent events and $A \in \mathcal{T}(\{A_n\})$. Then $P(A) \in \{0, 1\}$

The low for isomorphism degrees $\mathcal{C}$ are a Borel tailset and so satisfy Kolmogrov's $0-1$ law. Therefore it suffices to show that the complement has positive measure to ensure $\mu(\mathcal{C}) = 0$.

Graphs are universal objects

Theorem (Hirschfeldt, Khoussainov, Shore, Slinko)

There is an effective coding of an arbitrary countable structure $\mathcal{A}$ in a computable language into a countable directed graph $G(\mathcal{A})$ such that:

- $\mathcal{A} \cong \mathcal{B} \iff G(\mathcal{A}) \cong G(\mathcal{B})$
- *$\mathcal{A}$ is computable $\iff$ $G(\mathcal{A})$ is computable*
- *If $\mathcal{A}, \mathcal{B}$ are computable and for a turing degree d,* $\mathcal{A} \cong_d \mathcal{B} \iff G(\mathcal{A}) \cong_d G(\mathcal{B})$

Therefore to show that a degree d is low for isomorphism is equivalent to showing that for every pair of computable directed graphs G_0, G_1 there is a d computable isomorphism between G_0 and G_1 $\iff$ there is a computable one.

Proof of main theorem

We build two isomorphic computable directed graphs G and H and a Π^0_1 class $\mathcal{C} \subset 2^\omega$ such that

- $G \ncong_{\Delta^0_1} H$
- $\mu(\mathcal{C}) \geq 1/2$
- $X \in \mathcal{C} \implies X$ computes an isomorphism from $G \to H$.

Rest of the proof on the Black Board...

References

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Taking the path computably travelled
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The End

Questions? Comments?