A Brief Introduction to Modal Logic

Yuxiao Fu

University of Wisconsin-Madison

Logic Student Seminar, Fall 2022

Table of Contents

Table of Contents

Formulas

Similar to first-order logic, Modal Logic can be seen as an extension to propositional logic found useful in philosophy and linguistics. The language of **basic modal logic** is given by the following grammar:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \psi \vee \varphi \mid \Diamond \varphi$$

where p ranges over a given set of propositional variables. Next to the standard Boolean abbreviations $\top, \wedge, \rightarrow, \leftrightarrow$ we will also use $\Box := \neg \Diamond \neg$.

Readings

Examples

Modal logic can comes with different flavors depending on the reading of modal operators, three common readings are:

1. Possible: $\Diamond\varphi$ reads "It is possible that φ" and $\Box\varphi$ reads "It is necessarily that φ". Some truth include $\Box\varphi \to \Diamond\varphi$, $\varphi \to \Diamond\varphi$.
2. Episemic: $\Box\varphi$ reads "the agent knows that φ" and $\Diamond\varphi$ reads "the agent does not know that $\neg\varphi$". Note $\varphi \to \Box\varphi$ but not conversely.
3. Provable: $\Box\varphi$ reads "it is provable that φ" and $\Diamond\varphi$ reads "it is not provable that $\neg\varphi$". Löb formula states $\Box(\Box\varphi \to \varphi) \to \Box\varphi$.

Generalized

One may generalize the basic modal logic by adding more modal operators to propositional logic:

Definition

A **modal language** $\mathrm{ML}(\tau, \Phi)$ is built up using a modal similarity type $\tau = (O, \rho)$ and a set of proposition letters Φ, where nonempty O contains modal operators $\triangle_i$, and $\rho \colon O \to \mathbb{N}$ assigns each $\triangle_i$ its arity. The set $\mathrm{Form}(\tau, \Phi)$ of **modal formulas** over τ and Φ is given by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \triangle\left(\varphi_1, \ldots, \varphi_{\rho(\triangle)}\right)$$

where p ranges over Φ and $\triangle$ ranges over O.

Denote $\nabla(\varphi_1, \ldots, \varphi_n) := \neg \triangle(\neg\varphi_1, \ldots, \neg\varphi_n)$ for each $\triangle \in O$.

More Readings

Examples

1. Temporal: $O = \{\mathsf{F}, \mathsf{P}\}$. $\mathsf{F}\,\varphi$ reads "φ will happen at some Future time", and $\mathsf{P}\,\varphi$ reads "φ happened at some Past time". Their dual $\mathsf{G}\,\varphi$ and $\mathsf{H}\,\varphi$ reads "it is always Going to be φ" and "it always Has been φ". Some truth in this logic: $\mathsf{P}\,\phi \rightarrow \mathsf{GP}\,\phi$ ("whatever has happened will always have happened") and $\mathsf{F}\,\varphi \rightarrow \mathsf{FF}\,\varphi$.
2. Propositional Dynamic: Each diamond has the form $\langle\pi\rangle$, where π is a non-deterministic program, and $\langle\pi\rangle\varphi$ means "some terminating execution of π from the present state leads to a state bearing the information φ.' The dual $[\pi]\phi$ states that "every execution of π from the present state leads to a state bearing the information φ".

Kripke Models

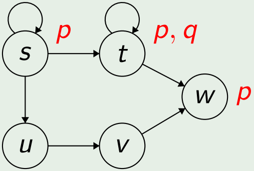
Definition

A **(Kripke) model** $\mathfrak{M} = (W, R, V)$ consists of a non-empty set W, a binary **accessibility relation** $R \subseteq W^2$ and a **valuation** $V\colon \Phi \to \mathcal{P}(W)$.

The underlying relational structure $\mathfrak{F} = (W, R)$ is a **(Kripke) frame**.

Examples

In the following model, $W = \{s, t, u, v, w\}$,
$R = \{\langle s, s\rangle, \langle s, t\rangle, \langle t, t\rangle, \langle t, w\rangle, \langle s, u\rangle, \langle u, v\rangle, \langle v, w\rangle\}$,
$V(p) = \{s, t, w\}$ and $V(q) = \{t\}$.



Kripke Semantics

Definition

Given a model $\mathfrak{M}$ we define the notion of a modal formula φ being **true** or **satisfied** in $\mathfrak{M}$ at a world $w \in \mathfrak{M}$, denoted $\mathfrak{M}, w \Vdash \varphi$, inductively by

$\mathfrak{M}, w \Vdash p$	iff	$w \in V(p)$
$\mathfrak{M}, w \Vdash \bot$	iff	never
$\mathfrak{M}, w \Vdash \neg\varphi$	iff	$\mathfrak{M}, w \nVdash \varphi$
$\mathfrak{M}, w \Vdash \varphi \vee \psi$	iff	$\mathfrak{M}, w \Vdash \varphi$ or $\mathfrak{M}, w \Vdash \psi$
$\mathfrak{M}, w \Vdash \Diamond \varphi$	iff	$\mathfrak{M}, v \Vdash \varphi$, for some $v \in W$ with Rwv.

A formula φ is **globally true** in a model $\mathfrak{M}$ if it is true at every $w \in \mathfrak{M}$, denoted $\mathfrak{M} \Vdash \varphi$; and **satisfiable in** $\mathfrak{M}$ if it is true in at least one $w \in \mathfrak{M}$.

Examples

$\mathfrak{M}, t \Vdash p \wedge q$, $\mathfrak{M}, t \Vdash \Diamond p$, $\mathfrak{M}, w \Vdash \neg q$, and $\mathfrak{M}, w \Vdash \Box \neg p$.

Satisfiability and Validity

Definition (for Models)

A formula φ is **satisfiable** if it is satisfiable in some model, and **valid** if it is globally true in every model.

Examples

$\Box \top$ is valid, $\Box p \wedge \Diamond \neg p$ is never satisfiable.

Definition (for Frames)

A formula φ is **valid in** $\mathfrak{F}$ if φ is globally true in model $(\mathfrak{F}, V)$ for every valuation V (denoted $\mathfrak{F} \Vdash \varphi$), and **valid in a class of frames** C if $\mathfrak{F} \Vdash \varphi$ for each $\mathfrak{F} \in$ C (denoted C $\Vdash \varphi$). A formula is **satisfiable in** $\mathfrak{F}$ if it is satisfiable in $(\mathfrak{F}, V)$ for some valuation V.

Examples

K $\Vdash \Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$, but not $\Diamond \Diamond p \to \Diamond p$.

Table of Contents

Modal Equivalence

Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models of the same modal similarity type τ, and let w and w' be states in $\mathfrak{M}$ and $\mathfrak{M}'$ respectively.

Definition

The τ-**theory** of w is $\{\varphi : \mathfrak{M}, w \Vdash \varphi\}$. w and w' are (modally) equivalent ($w \leftrightsquigarrow w'$) if they have the same τ-theories. $\mathfrak{M} \leftrightsquigarrow \mathfrak{M}'$ is defined similarly.

Bisimulations

Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be two models.

Definition

A non-empty binary relation $Z \subseteq W \times W'$ is called a **bisimulation** between $\mathfrak{M}$ and $\mathfrak{M}'$ ($Z : \mathfrak{M} \leftrightarrow \mathfrak{M}'$) if:

1. If wZw' then w and w' satisfy the same proposition letters.
2. If wZw' and Rwv, there exists $v' \in W'$ such that vZv' and $R'w'v'$.
3. If wZw' and $R'w'v'$, there exists $v \in W$ such that vZv' and Rwv.

If wZw' by bisimulation Z, we say w and w' is bisimilar ($w \leftrightarrow w'$).

A bisimulation is a relation between two models in which related states have identical atomic information and matching transition possibilities.

Examples

The following two models are bisimilar by

$$Z = \{(1, a), (2, b), (2, c), (3, d), (4, e), (5, e)\}.$$

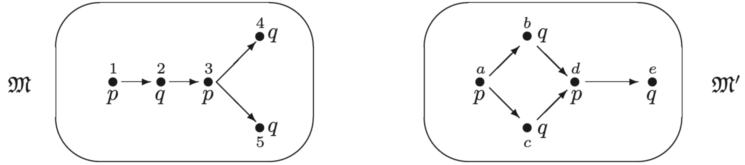


Fig. 2.4. Bisimilar models.

Invariance under Bisimulation

Let τ be a modal similarity type, and let $\mathfrak{M}, \mathfrak{M}'$ be τ-models.

Theorem

For every $w \in W$ and $w' \in W'$, $w \leftrightarrow w'$ implies that $w \rightsquigarrow w'$.

That is, modal formulas are invariant under bisimulation: modal formulas cannot distinguish between bisimilar states or between bisimilar models. This is different from first-order logic.

Examples

Observe that $\mathfrak{M}', a \Vdash \varphi(a)$ but $\mathfrak{M}, 1 \nVdash \varphi(1)$ for $\varphi(x)$:

$$\exists y_1 y_2 y_3 \left(y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge y_2 \neq y_3 \wedge Rxy_1 \wedge Rxy_2 \wedge Ry_1 y_3 \wedge Ry_2 y_3 \right).$$

So φ distinguishes $a \leftrightarrow 1$.

Hennessy-Milner

The converse does not hold in general, but does for image-finite models.

Definition

A τ-model $\mathfrak{M}$ is **image-finite** if for each state u and relation R, the set $\{(v_1, \ldots, v_n) : Ruv_1 \ldots v_n\}$ is finite.

Theorem (Hennessy-Milner)

Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two image-finite τ-models. Then, for every $w \in W$ and $w' \in W', w \leftrightarrow w'$ iff $w \leftrightsquigarrow w'$

Finite Models

Similar to compactness of first-order logic, modal logic has finite model property (FMP). For this, one needs the notion of filtration.

Definition

Let $\mathfrak{M} = (W, R, V)$ be a model and Σ a subformula closed set of formulas. Let $\leftrightsquigarrow_\Sigma$ be the equivalence relation on the states of $\mathfrak{M}$ defined by:

$$w \leftrightsquigarrow_\Sigma v \text{ iff } \forall\varphi \in \Sigma : \mathfrak{M}, w \Vdash \varphi \text{ iff } \mathfrak{M}, v \Vdash \varphi.$$

Denote $|w|_\Sigma$ the equivalence class of $w \in W$. Let $W_\Sigma = \{|w|_\Sigma : w \in W\}$. Suppose $\mathfrak{M}^f_\Sigma$ is any model $\left(W^f, R^f, V^f\right)$ such that:

1. $W^f = W_\Sigma$.
2. If Rwv then $R^f|w||v|$.
3. If $R^f|w||v|$ then for all $\Diamond\phi \in \Sigma$, if $\mathfrak{M}, v \Vdash \phi$ then $\mathfrak{M}, w \Vdash \Diamond\phi$.
4. $V^f(p) = \{|w| : \mathfrak{M}, w \Vdash p\}$, for all proposition letters p in Σ.

Then $\mathfrak{M}^f_\Sigma$ is called a **filtration** of $\mathfrak{M}$ through Σ.

Examples

Let $\mathfrak{M} = (\mathbb{N}, R, V)$, where $R = \{(0,1),(0,2),(1,3)\} \cup \{(n, n+1) : n \geq 2\}$, and V has $V(p) = \mathbb{N} - \{0\}$ and $V(q) = \{2\}$. For subformula closed $\Sigma = \{\Diamond p, p\}$. $\mathfrak{M}' = (\{|0|, |1|\}, \{(|0|, |1|), (|1|, |1|)\}, V')$, where $V'(p) = \{|1|\}$, is a filtration of $\mathfrak{M}$ through Σ.

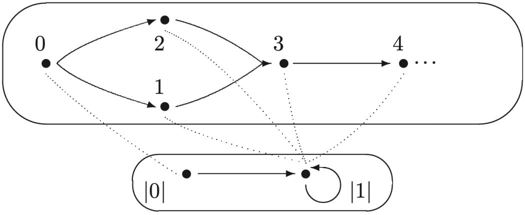


Fig. 2.6. A model and its filtration.

Filtration Theorem

By construction of the filtration, one has the following:

Theorem

Let $\mathfrak{M}^f = (W_\Sigma, R^f, V^f)$ be a filtration of $\mathfrak{M}$ through a subformula closed set Σ. Then for all formulas $\varphi \in \Sigma$, and all states $w \in W$, we have

$$\mathfrak{M}, w \Vdash \varphi \quad \textit{iff} \quad \mathfrak{M}^f, |w| \Vdash \varphi.$$

Finite Model Property is then realized under filtration:

Theorem (Finite Model Property)

If a basic modal formula φ is satisfiable, it is satisfiable on a finite model.

In fact, φ is satisfiable on a finite model containing at most 2^m nodes, where m is the number of subformulas of φ.

Standard Translation

One can link the modal logic to first-order logic in the following way:

Definition

For τ a modal similarity type and Φ a collection of proposition letters, let $L^1_\tau(\Phi)$ be the first-order language (with equality) which has unary predicates $P_0, P_1, P_2, \ldots$ corresponding to the proposition letters $p_0, p_1, p_2, \ldots$ in Φ, and an $(n+1)$-ary relation symbol R_Δ for each (n-ary) modal operator $\Delta \in O$.

Then we are ready for the translation.

Definition

Let x be a first-order variable. The **standard translation** ST_x taking modal formulas to first-order formulas in $L^1_\tau(\Phi)$ is defined as follows:

1. $\mathrm{ST}_x(p) = Px$.
2. $\mathrm{ST}_x(\bot) = x \neq x$.
3. $\mathrm{ST}_x(\neg\varphi) = \neg\,\mathrm{ST}_x(\varphi)$.
4. $\mathrm{ST}_x(\varphi \vee \psi) = \mathrm{ST}_x(\varphi) \vee \mathrm{ST}_x(\psi)$.
5. $\mathrm{ST}_x(\triangle\,(\varphi_1, \ldots, \varphi_n)) = \exists y_1 \ldots \exists y_n\,(R_\Delta x y_1 \ldots y_n \wedge \bigwedge_{i=1}^n \mathrm{ST}_{y_i}(\varphi_i))$ where $y_1, \ldots, y_n$ are new variables.

In case of basic modal logic, (5) becomes the usual quantifier

$$\mathrm{ST}_x(\Diamond\,\varphi) = \exists y\,(Rxy \wedge \mathrm{ST}_y(\varphi))\,, \quad \mathrm{ST}_x(\Box\,\varphi) = \forall y\,(Rxy \to \mathrm{ST}_y(\varphi))\,.$$

Examples

$$\mathrm{ST}_x(\Diamond(\Box\,p \to q)) = \exists y_1\,(Rxy_1 \wedge (\forall y_2\,(Ry_1 y_2 \to Py_2) \to Qy_1))\,.$$

Van Benthem's Theorem

This theorem precisely characterize the relation between first-order logic, modal logic, and bisimulations.

Definition

A first-order formula $\alpha(x)$ in $\mathcal{L}^1_\tau$ is **invariant for bisimulations** if whenever $\mathfrak{M}, w$ and $\mathfrak{M}', w'$ are two bisimilar models, then $\mathfrak{M} \vDash \varphi[w]$ iff $\mathfrak{M}' \vDash \varphi[w']$.

Theorem (van Benthem Characterization Theorem)

A first-order formula $\alpha(x)$ in $\mathcal{L}^1_\tau$ is invariant for bisimulations iff it is equivalent to the standard translation of a modal τ-formula.

Modal logic is the bisimulation invariant fragment of first-order logic.

Table of Contents

Frame Definability

Let φ a modal formula of similarity type τ, and F a class of τ-frames.

Definition

φ **defines** F if for all frames $\mathfrak{F}$, $\mathfrak{F}$ is in F iff $\mathfrak{F} \Vdash \varphi$. Similarly, if Γ is a set of modal formulas of this type, we say that Γ **defines** F if $\mathfrak{F}$ is in F iff $\mathfrak{F} \Vdash \Gamma$.

Theorem (Goldblatt-Thomason)

A first-order definable class F *of* τ*-frames is modally definable iff it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions.*

Automatic First-order Correspondence

Definition

Modal formula φ and first-order formula $\alpha(x)$ are called **local frame correspondents** of each other if for any frame $\mathfrak{F}$ and any state w of $\mathfrak{F}$:

$$\mathfrak{F}, w \Vdash \varphi \quad \text{iff} \quad \mathfrak{F} \models \alpha[w]$$

Modal formulas contains no proposition letters are **closed**, closed formulas have automatic first-order correspondence.

Theorem

Let φ be a closed modal formula, it is locally corresponds to a first-order formula $c_\varphi(x)$ which is computable from φ.

A modal formula is **uniform** if all its proposition letters occur uniformly. A proposition letter p occurs **uniformly** in a modal formula if it occurs only positively (in scope of even number of negations), or only negatively (in scope of odd number of negations).

Examples

$\Diamond(p \to q) = \Diamond(\neg p \vee q)$ is uniform for it is negative in p and positive in q.

Uniform formulas also have automatic first-order correspondence.

Theorem

Let φ be a uniform modal formula, it is locally corresponds to a first-order formula $c_\varphi(x)$ which is computable from φ.

Chagrova's Theorem

Does any modal formula φ has first-order Correspondence?

Theorem (Chagrova's Theorem)

It is noncomputable whether an arbitrary basic modal formula has a first-order correspondent.

However, there are computable subsets of modal formulas with first-order correspondents. One important example is Sahlqvist Formulas.

Sahlqvist Formulas

Definition

A **Sahlqvist antecedent** is built from $\bot, \top$, negative formulas and **boxed atom** $\Box^n p$ by applying $\Diamond$ and $\wedge$. A **Sahlqvist implication** is a modal formula of the form $\varphi \to \psi$, where φ is a Sahlqvist antecedent and ψ is a positive formula. A **Sahlqvist formula** is built from Sahlqvist implications by applying $\Box$ and $\vee$.

Theorem (Sahlqvist Correspondence)

For any Sahlqvist formula φ, there is a corresponding first-order sentence that holds in a frame iff φ is valid in the frame.

Sahlqvist-Van Benthem Algorithm

One can compute the first-order correspondence of Sahlqvist formulas. For simplicity, we demonstrate the algorithm for simple Sahlqvist Formulas (whose antecedent is built from only $\bot, \top$ and boxed atoms).

Definition (Sahlqvist-Van Benthem Algorithm)

1. Identify boxed atoms in the antecedent.
2. Draw the picture that discusses the minimal valuation that makes the antecedent true. Name the worlds involved by $t_0, \dots, t_n$.
3. Work out the minimal valuation i.e., get a first-order expression for it in terms of the named worlds.
4. Work out the standard translation of φ. Use the names you fixed for the variables that correspond to $\Diamond$'s in the antecedent.

Definition (Algorithm, cont.)

1. Pull out the quantifiers that bind t_i variables in the antecedent to the front. For this use the equivalences

$$\exists x\alpha(x) \wedge \beta \leftrightarrow \exists x(\alpha(x) \wedge \beta) \quad \exists x\alpha(x) \rightarrow \beta \leftrightarrow \forall x(\alpha(x) \rightarrow \beta)$$

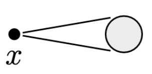
2. Replace all the predicates $P(x)$, $Q(x)$, etc., with the first-order expression corresponding to the minimal valuation.
3. Simplify, if possible.
4. Add $\forall x$ (binding the free variable of the standard translation) to the resulting first-order formula to obtain the global first-order correspondent.

If φ is a Sahlqvist formula, say $\Box(\varphi \rightarrow \psi) \vee \Box(\varphi' \rightarrow \psi')$ (where $\varphi \rightarrow \psi$ and $\varphi' \rightarrow \psi'$ are simple Sahlqvist formulas), then draw a diagram where outer $\Box$'s are treated as $\Diamond$'s and $\vee$ is treated as $\wedge$.

Examples

Examples

Let $\varphi = \Box p \to p$. The diagram



The minimal valuation is $V(p) = \{z : Rxz\}$. The standard translation of φ is $\forall y(Rxy \to P(y)) \to P(x)$. Replace $P(y)$ with Rxy and $P(x)$ with Rxx. We obtain $\forall y(Rxy \to Rxy) \to Rxx$. This is equivalent to Rxx. By adding $\forall x$ we obtain the global first-order correspondent $\forall xRxx$ (reflexivity).

Table of Contents

Normal Modal Logic

Definition

A **normal modal logic** Λ is a set of formulas that contains all tautologies, $\Box(p \to q) \to (\Box p \to \Box q)$, and $\Diamond p \leftrightarrow \neg \Box \neg p$, and closed under rules:

1. Modus ponens (MP): if φ and $\varphi \to \psi$, then ψ.
2. Uniform substitution (US): if φ, then θ, where θ is obtained from φ by replacing proposition letters in φ by arbitrary formulas.
3. Generalization (G): if φ, then $\Box \varphi$.

Denote the smallest normal modal logic **K**.

K turns out to be the 'minimal' system for reasoning about frames.

Instead of $\mathbf{K} \vdash \varphi$ one often writes $\vdash_{\mathbf{K}} \varphi$.

Examples

Here is derivation for $\vdash_{\mathbf{K}} \Box(A \wedge B) \rightarrow \Box A$:

$$\vdash_{\mathbf{K}} A \wedge B \rightarrow A \quad \text{(Propositional tautology)}$$
$$\vdash_{\mathbf{K}} \Box(A \wedge B \rightarrow A) \quad \text{(N)}$$
$$\vdash_{\mathbf{K}} \Box(A \wedge B \rightarrow A) \rightarrow (\Box(A \wedge B) \rightarrow \Box A) \quad (\mathbf{K}\text{-axiom})$$
$$\vdash_{\mathbf{K}} \Box(A \wedge B) \rightarrow \Box A \quad \text{(MP)}.$$

Soundness and Completeness for **K**

Fix a modal similarity type τ and a countable set Φ of proposition letters. Semantically, define the logic of a class of frames C to be the collection

$$\Lambda_C = \{\varphi \in \mathrm{Form}(\tau, \Phi) : C \Vdash \varphi\}.$$

Syntactically, the theorem of **K** is just **K**:

$$\mathrm{Th}(\mathbf{K}) = \{\varphi \in \mathrm{Form}(\tau, \Phi) : \mathbf{K} \vdash \varphi\} = \mathbf{K}.$$

We want them to coincide.

Theorem

A formula is a theorem of **K** *iff it is valid in every frames. i.e.:*

$$\mathbf{K} = \Lambda_F$$

Soundness is by easy induction yet completeness is harder.

Computability

Computability and Complexity naturally arises in normal modal logic. One important instance is the satisfiability/validity problems:

Definition (S-V Question)

Given a modal formula φ and a class of models M, is it computable whether φ is M-satisfiable/valid?

Observe φ is M-valid iff $\neg\varphi$ is not M-satisfiable, S is computable iff V is.

Harrop's theorem

Theorem

Every axiomatizable normal modal logic that has the finite model property with respect to an c.e. set of models M *is computable.*

Theorem (Harrop)

Every finitely axiomatizable normal modal logic with the finite model property is computable.

Check if a finite frame validates the axioms of Λ (finitely many).

Corollary

Being finitely axiomatizable and with FMP, **K** *is computable.*

Reference

Blackburn, Rijke, Venema. Modal Logic. *Cambridge U. Press.* 2001.