A GENTLE INTRODUCTION TO DESCENT

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1. GAGS Talk 1 – April 26, 2017

1.1. “Concrete” examples.

1.1.1. Let $X = \operatorname{Spec} R$ be an affine scheme and $\{U_i \to X\}$ a family of morphisms of affine schemes corresponding to the ring homomorphisms $\{R \to R_i\}$. Then any R-module M, thought of as a module over X, determines modules M_i over each U_i by extension of scalars $M_i = R_i \otimes_R M$.

$$\begin{tikzcd}
M_{i,j} \cong M_{j,i} \arrow[dr, squiggly] & & M_j \arrow[dr, squiggly] & \\
& U_i \times_X U_j \arrow[rr] \arrow[dd] & & U_j \arrow[dd] \\
M_i \arrow[dr, squiggly] & & M \arrow[dr, squiggly] & \\
& U_i \arrow[rr] & & X
\end{tikzcd}$$

The modules M_i are **“restrictions“** or **pullbacks** of M along $U_i \to X$, and are **compatible** in that if $M_{i,j}$ is the pullback of M_i over U_i along $U_i \times_X U_j \to U_i$, then we have isomorphisms $M_{i,j} \cong M_{j,i}$ that satisfy a coherence condition known as the **cocycle condition**.

We say that modules over affine schemes have the property of **descent** for a family $\{U_i \to X\}$ if any family of compatible modules M_i over U_i glues uniquely to a module M over X that the M_i are restrictions of.

Example 1.1.2. Do modules have descent for a Zariski principal open cover $\{U_i \to X\}$, i.e. a family corresponding to prinicpal localizations $\{R \twoheadrightarrow R_{f_i}\}$ so that $1 \in \langle f_i \rangle$?

Yes: the partition-of-unity argument from a first course on algebraic geometry shows that modules have descent with respect to Zariski principal open covers.

Example 1.1.3. Do modules have descent for a Zariski cover $\{U \to X\}$ corresponding to not necessarily principal localizations $\{R \twoheadrightarrow S_i^{-1}R\}$?

Yes, as long as the cover has a finite subcover; the proof is then word-for-word the same as the partition of unity argument for Zariski principal open covers. Recall that Zariski principal open covers always have finite subcovers, so this example is a strict generalization of the one above.

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This kind of descent is useful, e.g. in studying curves with finitely many singularities. For such a singular curve, this form of descent implies that to specify a quasi-coherent sheaf on the curve, one need only specify: 1) a quasi-coherent sheaf on the smooth locus, 2) modules over the stalks of the singularities, and 3) isomorphisms between their restrictions to the generic point(s).

Example 1.1.4. Do modules have descent for a Zariski cover $\{U \to X\}$ corresponding to flat ring homomorphisms $\{R \to R_i\}$?

Yes, as long as the cover has a finite subcover.

This kind of descent is the affine scheme version of **fpqc-descent**; fpqc covers of schemes are the natural extension of these finite flat covers of affine schemes to all schemes.

1.1.5. Descent theory is a categorical formalization of the notion of descent. It relies on the notion of fibrations which we'll explore in accordance with the following plan:

(1) "Concrete" examples
(2) Formalizing restrictions:
    - "Pseudofunctors" on a category $\mathcal{C} \xleftrightarrow{\text{Grothendieck construction}}$ (cloven) fibrations over the category $\mathcal{C}$
(3) Formalizing compatibility:
    - Representable "functors" $\mathrm{Hom}_{\mathcal{C}}(-, X) \longleftrightarrow$ representable fibrations $\mathcal{C}/X$ for objects $X \in \mathcal{C}$
    - Sieves over an object $X \in \mathcal{C}$, i.e. pre-composition stable classes $\mathcal{S}$ of morphisms to X = subfibrations $\mathcal{S} \hookrightarrow \mathcal{C}/X$
    - Category of fibered functors and fibered natural transformations between fibrations $[\mathcal{C}/X, \mathcal{F}] \xleftrightarrow{\text{Yoneda lemma}}$ Fiber $\mathcal{F}(X)$ of the target cloven fibration $\mathcal{F}$ over X
    - Category of **descent data** $\overset{\text{equiv.}}{\simeq}$ Category of fibered functors and fibered natural transforamtions between fibrations $[\mathcal{S}, \mathcal{F}]$.
(4) Descent conditions for a sieve $\mathcal{S}_{\mathcal{U}}$ generated by a family $\mathcal{U} = \{U_i \to X\}$:
    - Fibrations generalize presheaves.
    - **Prestacks** are fibrations $\mathcal{F}$ such that the induced functor $[\mathcal{S}_{\mathcal{U}}, \mathcal{F}] \leftarrow [\mathcal{C}/X, \mathcal{F}]$ is fully faithful; prestacks generalize uniqueness of gluing condition of separated presheaves.
    - **Stacks** are fibrations $\mathcal{F}$ such that the induced functor $[\mathcal{S}_{\mathcal{U}}, \mathcal{F}] \leftarrow [\mathcal{C}/X, \mathcal{F}]$ is fully faithful and essentially surjective; stacks generalize the uniqueness of gluing and existence of gluing conditions of sheaves.

This development is mostly distilled from Part I of FGA Explained.

1.2. **Formalizing restrictions: pseudofunctors, the Grothendieck construction, fibrations.**

Definition 1.2.1. Given a category $\mathcal{C}$, a **"pseudofunctor"** $\mathcal{F}$ associates

(1) To each object $X \in \mathcal{C}$ a category $\mathcal{F}(X)$
(2) To each morphism $X \xrightarrow{f} Y \in \mathcal{C}$ a **pullback functor** $\mathcal{F}(X) \xleftarrow{f^*} \mathcal{F}(Y)$.
(3) For each object X, a natural isomorphism $\mathrm{id}_X^* \cong \mathrm{id}_{\mathcal{F}(X)}$
(4) To each pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ a natural isomorphism $(g \circ f)^* \cong f^* g^*$

satisfying the **coherence axioms** asserting that the following pairs of natural isomorphisms between (composites of) pullback functors coincide:

- $f^*\mathrm{id}_X^* \cong f^*\mathrm{id}_{\mathcal{F}(X)} = f^*$ and $f^*\mathrm{id}_X^* \cong (\mathrm{id}_X \circ f)^* = f^*$
- $\mathrm{id}_X^* g^* \cong \mathrm{id}_{\mathcal{F}(X)} g^* = g^*$ and $\mathrm{id}_X^* g^* \cong (g \circ \mathrm{id}_X)^* = g^*$
- $f^*g^*h^* \cong f^*(h \circ g)^* \cong (h \circ g \circ f)^*$ and $f^*g^*h^* \cong (g \circ f)^*h^* \cong (h \circ g \circ f)^*$

Example 1.2.2. For $\mathcal{C}$ the category of affine scheme, we have a "pseudofunctor" associating to each affine scheme $X = \operatorname{Spec} R$ the category of R-modules, and to each morphism $X' \to X$ corresponding to a ring homomorphism $R \to R'$, the extension-of-scalars functors $R'\text{-}\mathcal{M}od \xleftarrow{-\otimes_R R'} R\text{-}\mathcal{M}od$. The natural isomorphisms between composites of pullback functors are the natural isomorphisms between itereated tensor products; in particular, their coherence follows from the coherence of the natural isomorphisms between iterated tensor products.

Remark 1.2.3. "Pseudofunctor" is in quotes because its values can be large categories (like the category of R-modules), and there is no category of all large categories unless you adopt a stronger foundation such as Grothendieck universes. Nevertheless, even without Grothendieck universes, "pseudofunctor" is just as well-defined a notion as the notion of a large category, but you have to dig into first-order logic to understand how.

The "pseudo" in "pseudofunctor" refers to the fact that the association of morphisms to pullback functors is only functorial up to an explicit choice of natural isomorphisms; we would have merely a "functor" in the case where the natural isomorphisms are all identities.

Definition 1.2.4 (Grothendieck construction)**.** Given a "pseudofunctor" $\mathcal{F}$, we construct a category (abusively also labeled $\mathcal{F}$) and a functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$ as follows:

1. (1) The collection of objects of $\mathcal{F}$ is the disjoint union of the collections of objects of $\mathcal{F}(X)$ for each $X \in \mathcal{C}$; the functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$ sends an object $A \in \mathcal{F}(X)$ to $X \in \mathcal{C}$.
2. (2) A morphism $A \xrightarrow{\alpha} B \in \mathcal{F}$ from $A \in \mathcal{F}(X)$ to $B \in \mathcal{F}(Y)$ consists of a pair of morphisms $X \xrightarrow{f} Y \in \mathcal{C}$ and $f^*B \xleftarrow{a} A \in \mathcal{F}(X)$; the functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$ sends a morphism $A \xrightarrow{(f,a)} B$ to $X \xrightarrow{f} Y$.
3. (3) The composite $A \xrightarrow{(f,a)} B \xrightarrow{(g,b)} C$ is given by $A \xrightarrow{(g \circ f, c)} C$ where the vertical morphism $(g \circ f)^*C \xleftarrow{c} A$ is given by the composite $(g \circ f)^*C \cong f^*g*^C \xleftarrow{f^*b} B \xleftarrow{a} A$.

Exercise 1.2.5. Check that the Grothendieck construction does indeed produce a category. In other words, verify that composition is associative and that the pair $(X \xrightarrow{\mathrm{id}_X} X, A \cong \mathrm{id}_X^* A)$ is an identity morphism for $A \in \mathcal{F}$; you will have to use the coherence axioms satisfied by the natural isomorphisms of the "pseudofunctor".

Remark 1.2.6. The vertical fiber $p^{-1}(X)$ of objects mapping to X and morphisms mapping to $X \xrightarrow{\mathrm{id}_X} X$ is isomorphic to the category $\mathcal{F}(X)$. The **total category** $\mathcal{F}$ associated to a "pseudofunctor" has "vertical" morphisms that are the same as the values of the "pseudofunctor" and "horizontal" morphisms are the fewest morphisms that could come from morphisms in $\mathcal{C}$ in a sense we make precise below.

Definition 1.2.7. A functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$ is a **fibration** if for every morphism $X \xrightarrow{f} Y$ and every object $B \in p^{-1}(Y)$, there is a morphism $A \xrightarrow{\alpha} B$ in $\mathcal{F}$ such that

(1) α is a **lift** of $X \xrightarrow{f} Y$ in the sense that $p\alpha = f$.
(2) $A \xrightarrow{\alpha} B$ is p**-cartesian** in the sense that whenever the projection $pA' \xrightarrow{p\beta} pB = Y$ of a morphism $A' \xrightarrow{\beta} B$ factors as $pA' \xrightarrow{g} pA \xrightarrow{p\alpha} pB$, $A' \xrightarrow{\beta} B$ factors uniquely as $A' \xrightarrow{\gamma} A \xrightarrow{\alpha} B$ with $p\gamma = g$.
    In other words, α is p-cartesian if factorizations of p-images through the p-image of α lift uniquely to factorizations through α.

A **cloven fibration** is a fibration with a **cleavage**, that is, a choice of p-cartesian lifts $f^*B \to B \in \mathcal{F}$ for each pair of a morphism $X \xrightarrow{f} Y \in \mathcal{C}$ and an object $B \in p^{-1}(Y)$.

Remark 1.2.8. As worked out in the exercies below, the Grothendieck construction produces a cloven fibration out of a pseudofunctor, and inversely every cleavage of a fibration determines a "pseudofunctor". The advantage of fibrations over "pseudofunctors" is that the structure of natural isomorphisms between pullback functors is automatically induced and kept track of by the universal properties of p-cartesian morphisms. Consequently, the theory of fibrations is simpler to describe and develop than the corresponding theory of "pseudofunctors".

Example 1.2.9. Consider the codomain functor $\mathcal{C} \xleftarrow{\mathrm{cod}} \mathcal{C}^{\to}$ be the category whose objecsts are morphisms $A \to X \in \mathcal{C}$, and whose morphisms are commutative squares. A lift of a morphism $X \xrightarrow{f} Y \in \mathcal{C}$ along an object $Y \xleftarrow{b} B$ in $\mathrm{cod}^{-1}(Y)$ is an object $X \xleftarrow{a} A$ in $\mathrm{cod}^{-1}(X)$ together with a morphism given by a commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "a"] & B \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

This lift will be cod-cartesian if whenever we have a pair of morphisms $X' \xleftarrow{a'} A' \xrightarrow{\beta} A$ that form a commutative square with $X' \xrightarrow{g} X \xrightarrow{f} Y \xleftarrow{b} B$, there is a unique morhism $A' \xrightarrow{\gamma} A$ so that $A' \xrightarrow{\beta} A$ factors as $A' \xrightarrow{\beta} A \xrightarrow{\alpha} B$ and the square formed by $X' \xleftarrow{a'} A' \xrightarrow{\gamma} A$ and $X' \xrightarrow{g} X \xleftarrow{a} A$ commutes.

$$\begin{tikzcd} A' \arrow[r, dashed, "\gamma"] \arrow[rr, bend left, "\beta"] \arrow[d, "a'"] & A \arrow[r, "\alpha"] \arrow[d, "a"] & B \arrow[d, "b"] \\ X' \arrow[r, "g"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

By taking $g = \mathrm{id}_X$, we see that cod-cartesian squares are pullback squares, and conversely pullback squares are cod-cartesian by considering the factorization of the squares formed by $X \xleftarrow{g \circ a'} A \xrightarrow{\beta} B$ and $X \xrightarrow{f} Y \xleftarrow{b} B$. In particular, the codomain functor $\mathcal{C} \xleftarrow{\mathrm{cod}} \mathcal{C}^{\to}$ is a fibration if and only if $\mathcal{C}$ has pullbacks; a cleavage for $\mathcal{C} \xleftarrow{\mathrm{cod}} \mathcal{C}^{\to}$ is a choice of pullback squares for any pair of morphisms $X \xrightarrow{f} Y \xleftarrow{a} A$.

Exercise 1.2.10. The pullback lemma from basic category theory states that when the right-hand square α in the commutativie diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

is a pullback square, the the left-hand square α is a pullback square if and only if the whole rectangle $\beta \circ \alpha$ is a pullback square.

Verify that this is a general propery of p-cartesian morphisms for any functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$, i.e. if $B \xrightarrow{\beta} C$ is p-cartesian, then $A \xrightarrow{\alpha} B$ is p-cartesian if and only if the composite $\beta \circ \alpha$ is p-cartesian.

Exercise 1.2.11. The functor $\mathcal{C} \xleftarrow{p} \mathcal{F}$ produced by the Grothendieck construction is a fibration with cleavage given by $f^*B \xrightarrow{(f, \mathrm{id}_{f^*B})} B$ for any $B \in p^{-1}(Y)$ and $X \xrightarrow{f} Y$.

Reversely, given a morphism $X \xrightarrow{f} Y$, a choice of p-cartesian lifts $f^*B \to B$ for each $B \in p^{-1}(Y)$ extends by the universal propery of p-cartesian morphisms to a pullback functor $p^{-1}(X) \xleftarrow{f^*} p^{-1}(Y)$. Furthermore, and again by the universal property of p-cartesian morphisms, a cleavage for a fibration also determines the requisite coherent natural isomorphisms between composites of the pullback functors just described.

2. GAGS Talk 2 - May 3, 2017

2.1. **Formalizing compatibility: representible fibrations and the Yoneda lemma.**

Example 2.1.1. Consider the domain functor $\mathcal{C} \xleftarrow{\mathrm{dom}} \mathcal{C}^{\to}$. A lift of a morphism $A \xrightarrow{f} B \in \mathcal{C}$ along an object $B \xrightarrow{b} Y$ in $\mathrm{dom}^{-1}(B)$ is an object $X \xrightarrow{a} A$ in $\mathrm{dom}^{-1}(X)$ together with a morphism given by a commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\
X \arrow[r, "\alpha"] & Y
\end{tikzcd}$$

Given a morphism $A \xrightarrow{f} B$ and an object $B \xrightarrow{b} Y$ in $\mathrm{dom}^{-1}(B)$, a lift is a pair of morphism $X \xrightarrow{a} A \xrightarrow{\alpha} B$ so that the together with $X \xrightarrow{f} Y \xrightarrow{b} B$ they form a commutative square.

This lift is dom-cartesian if for any pair of morphisms $A' \xrightarrow{a'} X' \xrightarrow{\beta} Y$ determining a commutative square with $A' \xrightarrow{g} A \xrightarrow{f} B \xrightarrow{b} Y$, there is a unique morphism $X' \xrightarrow{\gamma} X$ through which $X' \xrightarrow{\beta} Y$ factors as $X' \xrightarrow{\gamma} X \xrightarrow{\alpha} Y$, and such that the square determined by $A' \xrightarrow{a'} X' \xrightarrow{\gamma} X$ and $A' \xrightarrow{g} A \xrightarrow{a} X$ commutes.

$$\begin{tikzcd}
A' \arrow[r, "g"] \arrow[d, "a'"] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\
X' \arrow[r, dashed, "\gamma"] \arrow[rr, bend right, "\beta"'] & X \arrow[r, "\alpha"] & Y
\end{tikzcd}$$

Evidently, $A \xrightarrow{b \circ f} Y \xrightarrow{\mathrm{id}_Y} Y$ determines a dom-cartesian lift. In particular, $\mathcal{C} \xleftarrow{\mathrm{dom}} \mathcal{C}^{\rightarrow}$ is always a cloven fibration; the "pseudofunctor" determined by the cleavage that sends $X \mapsto X/\mathcal{C}$ with pullback functors associated to $X \xrightarrow{f} Y$ given by pre-composition $X/\mathcal{C} \xleftarrow{f^*} Y/\mathcal{C}$ (where $X/\mathcal{C} = \mathrm{dom}^{-1}(X)$ is the category whose objects are morphisms out of X and whose morphisms are commutative triangles).

Example 2.1.2. Consider the functor $\mathcal{C} \xleftarrow{\mathrm{dom}} \mathcal{C}/X$ where $\mathcal{C}/X$ is the category whose objects are morphisms $A \to B$ and whose morphisms are commutative triangles
$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "b \circ f"'] & & B \arrow[dl, "b"] \\
& X &
\end{tikzcd}$$
. Each morphism in $\mathcal{C}/X$ is the unique dom-cartesian of lift of $A \xrightarrow{f} B \in \mathcal{C}$ along the object $B \xrightarrow{b} X$ of $\mathcal{C}/X$.

Thus $\mathcal{C} \xleftarrow{\mathrm{dom}} \mathcal{C}/X$ is a fibration. Furthermore, its unique cleavage corresponds to the representable "functor" sending $A \in \mathcal{C}$ to the class (i.e. discrete category) $\mathrm{Hom}_{\mathcal{C}}(A, X)$, and each $A \xrightarrow{f} B \in \mathcal{C}$ to the pre-composition class function $\mathrm{Hom}_{\mathcal{C}}(A, X) \xleftarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(B, X)$.

Definition 2.1.3. For each object $X \in \mathcal{C}$, the fibration $\mathcal{C} \xleftarrow{\mathrm{dom}} \mathcal{C}/X$ is called a **representable fibration**.

Remark 2.1.4. The particular choice of cleavages via pre-composition for the dom-cartesian lifts of the domain fibration and for the representable fibrations are such that the natural isomorphisms of the associated "pseudofunctors" are all identities. Such a cleavage of a fibration is called a **splitting**, and the Grothendieck construction actually establishes a correspondence between split fibrations and "functors".

Example 2.1.5. Suppose $\mathcal{S}$ is a class of morphisms into $X \in \mathcal{C}$ that is stable under precomposition in the sense that if $A \xrightarrow{h} X$ factors as $A \xrightarrow{g} B \xrightarrow{f} X$ for $B \xrightarrow{f} X \in \mathcal{S}$, then $A \xrightarrow{h} X \in \mathcal{S}$. By the Grothendieck construction, such a class $\mathcal{S}$ corresponds to a (split) subfibration $\mathcal{S} \hookrightarrow \mathcal{C}/X \xrightarrow{\mathrm{dom}} \mathcal{C}$.

Definition 2.1.6. The **sieve $\mathcal{S}_{\mathcal{U}}$ on** X generated by a family of morphisms $\mathcal{U} = \{U_i \to X\} \in \mathcal{C}$ is the fibration corresponding to the pre-composition-stable class $\{A \xrightarrow{h} X \in \mathcal{C} : A \xrightarrow{h} X$ factors as $A \xrightarrow{g} B \xrightarrow{f} X$ for some $B \xrightarrow{f} X \in \mathcal{U}\}$.

Definition 2.1.7. Given two fibrations $\mathcal{C} \xleftarrow{p_1} \mathcal{F}$ and $\mathcal{C} \xleftarrow{p_2} \mathcal{G}$, a functor $\mathcal{F} \xrightarrow{a} \mathcal{G}$ is **fibered** if the triangle

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "a"] \arrow[dr, "p_1"'] & & \mathcal{G} \arrow[dl, "p_2"] \\ & \mathcal{C} & \end{tikzcd}$$

commutes and $\mathcal{F} \xrightarrow{a} \mathcal{G}$ sends p_1-cartesian morphisms to p_2-cartesian morphisms.

A natural transformation $a \overset{\alpha}{\Rightarrow} b\colon \mathcal{F} \to \mathcal{G}$ between fibered functors is itself **fibered** if the triangle

$$\begin{tikzcd} \mathcal{F} \arrow[rr, bend left, "a"] \arrow[rr, bend right, "b"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[dr, "p_1"'] & & \mathcal{G} \arrow[dl, "p_2"] \\ & \mathcal{C} & \end{tikzcd}$$

commutes in the sense that $p_1 = p_2 a \xRightarrow{p_2\alpha} p_2 b = p_1 \colon \mathcal{F} \to \mathcal{C}$ is the identity natural transformation.

Lemma 2.1.8 (Yoneda lemma)**.** *The evaluation functor $[\mathcal{C}/X, \mathcal{F}] \to \mathcal{F}(X) = p^{-1}(X)$ sending a fibered natural transformation $a \overset{\alpha}{\Rightarrow} b$ to the component $a(X \xleftarrow{\mathrm{id}_X} X) \xrightarrow{\alpha_{X \xleftarrow{\mathrm{id}_X} X}} b(X \xleftarrow{\mathrm{id}_X} X)$ is fully faithful, i.e. bijective on morphisms.*

$$\begin{tikzcd} a(X \xleftarrow{f} Y) = a(f^*(X \xleftarrow{\mathrm{id}_X} X)) \arrow[r] \arrow[d, dashed, "\alpha_{X \xleftarrow{f} Y}"] & a(X \xleftarrow{\mathrm{id}_X} X) \arrow[d, "\alpha_{X \xleftarrow{\mathrm{id}_X} X}"] \\ b(X \xleftarrow{f} Y) = b(f^*(X \xleftarrow{\mathrm{id}_X} X)) \arrow[r] & b(X \xleftarrow{\mathrm{id}_X} X) \end{tikzcd}$$

Furthermore, each cleavage of the fibration $\mathcal{C} \xleftarrow{p} \mathcal{F}$ determines a section (up to a natural isomorphism) $p^{-1}(X) \hookrightarrow [\mathcal{C}/X, \mathcal{F}]$ of the evaluation functor given on morphisms by

$$\begin{tikzcd} (f \circ g)^* a(X \xleftarrow{\mathrm{id}_X} X) = a'(Z \xleftarrow{f \circ g} X) \arrow[d, dashed, "a'(Z \xrightarrow{g} Y)"] \arrow[dr] & \\ f^* a(X \xleftarrow{\mathrm{id}_X} X) = a'(X \xleftarrow{f} Y) \arrow[r] & a(X \xleftarrow{\mathrm{id}_X} X) \end{tikzcd}$$

In particular, each cleavage determines an equivalence of categories $p^{-1}(X) = \mathcal{F}(X) \simeq [\mathcal{C}/X, \mathcal{F}]$.

Exercise 2.1.9. Verify the Yoneda lemma. Explicitly, show that

(1) the vertical morphisms $a(X \xleftarrow{f} Y) \xrightarrow{\alpha_{X \xleftarrow{f} Y}} b(X \xleftarrow{f} Y)$ induced by the combination of the universal property of the p-cartesian (since the functor is fibered) morphism $b(X \xleftarrow{f} Y) = b(f^*(X \xrightarrow{\mathrm{id}_X} X)) \to b(X \xleftarrow{\mathrm{id}_X} X)$ over the morphism $Y \xrightarrow{f} Y$ in $\mathcal{C}$, and the factorization $Y \xrightarrow{\mathrm{id}_Y} Y \xrightarrow{f} X$ of the p-image $Y \xrightarrow{f} X \xrightarrow{\mathrm{id}_X} X$ of the composite $a(X \xleftarrow{f} Y) = a(f^*(X \xrightarrow{\mathrm{id}_X} X)) \to a(X \leftarrow \mathrm{id}_X X) \xrightarrow{\alpha_{X \xleftarrow{\mathrm{id}_X} X}} b(X \xleftarrow{\mathrm{id}_X} X)$ in $\mathcal{F}$ assemble into a natural transformation $a \overset{\alpha}{\Rightarrow} b$;

(2) defining $a'(Z \xrightarrow{g} Y)$ for a morphism
$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "f\circ g"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$
in $\mathcal{C}$ as the induced morphism by the combination of the universal property of the p-cartesian $f^*(a(X \xleftarrow{\mathrm{id}_X} X) \to a(X \xleftarrow{\mathrm{id}_X} X)$ over $Y \xrightarrow{f} X$ and the factorization $Z \xrightarrow{g} Y \xrightarrow{f} X$ of the p-image of the p-cartesian morphism $(f \circ g)^* a(X \xleftarrow{\mathrm{id}_X} X) \to a(X \xleftarrow{\mathrm{id}_X} X)$ determines a fibered functor $\mathcal{C}/X \xrightarrow{a'} \mathcal{F}$ together with an isomorhism $a'(X \xleftarrow{\mathrm{id}_X} X) \cong a(X \xleftarrow{\mathrm{id}_X} X)$ in $p^{-1}(X) = \mathcal{F}(X)$.

(3) Whenever a fully faithful functor has a section up to an isomorphism on objects, the section extends uniquely to section functor up to a natural isomorphism.

Remark 2.1.10. The usual version of the Yoneda lemma seen in basic category theory courses is the special case where $\mathcal{C} \xleftarrow{p} \mathcal{F}$ is a (small) discrete fibration, i.e. where each fiber $p^{-1}(X)$ is a (small) category whose only morphisms are identities.

The proof outlined in the above exercise then reduces to showing 1) that the evaluation functor is injective (since a fully faithful functor to a discrete category is the same as a functor injective on objects from a discrete category), and 2) that the (necessarily unique) cleavage of the discrete fibration determines an actual section of the (now injective) evaluation functor. This produces the usual isomorphism between the discrete category $\mathrm{Hom}_{\mathcal{C}}(-, X) \cong \mathcal{F}(X) = p^{-1}(X)$ since an injective functor with a section is an isomorphism.

2.2. **Formalizing compatibility.**

Definition 2.2.1. Given a family of morphisms $\mathcal{U} = \{U_i \to X\}$ in X and a (cloven) fibration $\mathcal{C} \xleftarrow{p} \mathcal{F}$, a category of **descent data for** $\mathcal{U}$ is a category equivalent to the category of fibered functors and fibered natural transformations $[\mathcal{S}_{\mathcal{U}}, \mathcal{F}]$ where $\mathcal{S}_{\mathcal{U}}$ is the sieve generated by $\mathcal{U}$.

Example 2.2.2. $[\mathcal{S}_{\mathcal{U}}, \mathcal{F}]$ is itself a category of descent data for $\mathcal{U}$. Explicitly,

(1) a **descent datum** in this category assigns an object $M_f \in \mathcal{F}(V) = p^{-1}(V)$ to each morphism $V \xrightarrow{f} X$ that factors as $V \to U_i \to X$ (i.e. to each morphism in the sieve), together with a choice of **compatibility isomorphisms** $M_f \cong f_i^* M_i$ in each $\mathcal{F}(V) = p^{-1}(V)$ for each factorization $V \xrightarrow{f_i} U_i \to X$ of $V \xrightarrow{f} X$, where $M_i \in \mathcal{F}(U_i) = p^{-1}(U_i)$ is the object assigned to $U_i \to X$.

(2) a **morphism of descent data** is a family of morphisms $M_f \xrightarrow{\alpha_f} N_f$ in $\mathcal{F}(V) = p^{-1}(V)$ for each $V \xrightarrow{f} X$ in the sieve, so that for each factorization $V \xrightarrow{f_i} U_i \to X$ of $V \xrightarrow{f} X$ the diagrams
$$\begin{tikzcd} M_f \arrow[r, "\cong"] \arrow[d, "\alpha_f"] & f_i^* M_i \arrow[d, "f_i^* \alpha_i"] \\ N_f \arrow[r, "\cong"] & f_i^* N_i \end{tikzcd}$$
commute.

Example 2.2.3. Another category of descent data for $\mathcal{U}$ is given as follows.

(1) A **descent datum** consists of a choice of objects $M_i \in \mathcal{F}(U_i) = p^{-1}(U_i)$ for each $U_i \to X$ in $\mathcal{U}$, plus a choice of **transfer isomorphisms** $f_i^* M_i \cong f_j^* M_j$ in $p^{-1}(V) \cong \mathcal{F}(V)$ for each commutative square
$$\begin{tikzcd} V \arrow[r, "f_i"] \arrow[d, "f_j"'] & U_i \arrow[d] \\ U_j \arrow[r] & X \end{tikzcd},$$
subject to the **cocycle condition** that the composite $f_i^* M_i \cong f_j^* M_j \cong f_k^* M_k$ is the isomorphism $f_i^* M_i \cong f_k^* M_k$

(2) A **morphism of descent data** is a family of morphisms $M_i \xrightarrow{\alpha_i} N_i$ so that the diagrams
$$\begin{tikzcd} f_i^* M_i \arrow[r, "\cong"] \arrow[d, "f_i^* \alpha_i"] & f_j^* M_j \arrow[d, "f_j^* \alpha_j"] \\ f_i^* N_i \arrow[r, "\cong"] & f_j^* N_j \end{tikzcd}$$
commute for every commutative square
$$\begin{tikzcd} V \arrow[r, "f_i"] \arrow[d, "f_j"'] & U_i \arrow[d] \\ U_j \arrow[r] & X \end{tikzcd}.$$

Example 2.2.4. Suppose that $\mathcal{C}$ is equipped with a choice of pullback cubes

$$\begin{tikzcd} & U_{i,j,k} \arrow[rr, "\pi^i"] \arrow[dl, "\pi^k"'] \arrow[dd, "\pi^j"] & & U_{j,k} \arrow[dl, "\pi_j"] \arrow[dd, "\pi_k"] \\ U_{i,j} \arrow[rr, "\pi_j"] \arrow[dd, "\pi_i"'] & & U_j \arrow[dd] & \\ & U_{i,k} \arrow[rr, "\pi_k"] \arrow[dl, "\pi_i"'] & & U_k \arrow[dl] \\ U_i \arrow[rr] & & X & \end{tikzcd}$$

for each triple of morphisms $U_i, U_j, U_k \to X$. Then the following is a category of descent data for $\mathcal{U}$:

(1) A **descent datum** consists of objects $M_i \in \mathcal{F}(U_i) = p^{-1}(U_i)$ together with **transition isomorphisms** $\phi_{i,j} \colon M_{i,j} \cong M_{j,i}$ where $M_{i,j} = \pi_i^* M_i$ for the pullback morphism $U_i \times U_j \xrightarrow{\pi_j} U_j$, and subject to the **cocycle condition** that the composite $M_{i,k,j} \cong M_{i,j,k} \overset{\pi^{k,*}\phi_{i,j}}{\cong} M_{j,i,k} \cong M_{j,k,i} \overset{\pi^{i,*}\phi_{j,k}}{\cong} M_{k,j,i} \cong M_{k,i,j}$ is the isomorphism $M_{i,k,j} \overset{\pi^{j,*}\phi_{i,j}}{\cong} M_{k,i,j}$, where $M_{i,j,k} = \pi^{k,*}\pi_i^* M_i$.

(2) A **morphism of descent data** is a family of morphisms $M_i \xrightarrow{\alpha_i} N_i$ compatible with the transfer isomorphisms in the sense that the square

$$\begin{tikzcd} M_{i,j} \arrow[r, "\cong", "\phi_{i,j}"'] \arrow[d, "\pi_i^* \alpha_i"'] & M_{j,i} \arrow[d, "\pi_j^* \alpha_j"] \\ N_{i,j} \arrow[r, "\cong", "\psi_{i,j}"'] & N_{j,i} \end{tikzcd} .$$

Exercise 2.2.5. Verify that the functors from $[\mathcal{S}_{\mathcal{U}}, \mathcal{F}]$ to each of the above two categories of descent data are fully faithful and essentially surjective.

Definition 2.2.6. Given a family $\mathcal{U}$ of morphisms $\{U_i \to X\}$ in a category $\mathcal{C}$, and a (cloven) fibration $\mathcal{C} \xleftarrow{p} \mathcal{F}$, we say that the fibration is

(1) a **prestack** if the natural functor $[\mathcal{S}_{\mathcal{U}}, \mathcal{F}] \leftarrow [\mathcal{C}/X, \mathcal{F}]$ induced by the (fibered) inclusion $\mathcal{S}_{\mathcal{U}} \hookrightarrow \mathcal{C}/X$ is fully faithful;
(2) a **stack** if the natural functor $[\mathcal{S}_{\mathcal{U}}, \mathcal{F}] \leftarrow [\mathcal{C}/X, \mathcal{F}]$ is both fully faithful and essentially surjective.

Example 2.2.7. When $\mathcal{C} \xleftarrow{p} \mathcal{F}$ is a discrete fibration, we can think of the associated "functor" $\mathcal{F}$ as a "class"-valued presheaf. Then being fully faithful corresponds to the comparison functor being injective, and hence to the presheaf being separated. On the other hand, being fully faithful and essentially surjective corresponds to the comparision functor being bijective, and hence to the presheaf being a sheaf.