HOMOLOGICAL ALGEBRA (MATH 750)
PROBLEMS AND EXERCISES

Conventions

The problems below come in three types: Exercises, Problems, and Challenges.

Exercises are supposed to be relatively straightforward, but could be technical. Typically, they would involve verification of some properties that I consider important, but insufficiently interesting for the class. They are also supposed to make sure that you are capable of operating with the ideas of this class. If you are convinced that you understand what is involved in an exercise, there is probably no reason to work out all the details (but you'd better be sure!). On the other hand, if you don't see how to solve an exercise, it is a sign that you may be missing something important, and you should ask about this as soon as possible. Please do not hand the exercises in.

Problems are supposed to be more enlightening, and appropriate as homework problems. (Although I have not decided what fraction of them to assign.)

Challenges are questions that I am not sure one can answer with information available at this point. Privately, I think of Challenges as problems of 'Chuck Norris' difficulty. It still may be a good idea to try them out, if only to understand why they are hard (or perhaps to find out that I am missing something simple and they are not hard at all!) If you know more advanced homological algebra (that is, some of the things that were not covered in class yet), you may have tools to solve these challenges; otherwise, it may be a good idea to come back to them later in the course.

1. Derived functors

1.1. Ext **for abelian groups.** By default, all 'objects' in this section are abelian groups, Hom refers to the group of homomorphisms between abelian groups, etc.

Exercise 1.1.1. In class, we verified that for any $M \in \mathrm{Ab}$, the functor $\mathrm{Hom}(M, -)$ is left exact. Verify that for any $N \in \mathrm{Ab}$, the functor $\mathrm{Hom}(-, N)$ is left-exact as well.

Exercise 1.1.2. In class, we verified that for any $M \in \mathrm{Vect}$, the functor $\mathrm{Hom}(M, -)$ is exact. In other words, if

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

is a short exact sequence of vector spaces, then the induced sequence

$$0 \to \mathrm{Hom}(M, N_1) \to \mathrm{Hom}(M, N_2) \to \mathrm{Hom}(M, N_3) \to 0$$

is exact as well. Verify the following equivalent formulations of this exactness:

- For any short exact sequence

$$N_1 \to N_2 \to N_3,$$

the sequence

$$\mathrm{Hom}(M, N_1) \to \mathrm{Hom}(M, N_2) \to \mathrm{Hom}(M, N_3)$$

is exact.

- For any complex $C^\bullet$, consider the complex $\mathrm{Hom}(M, C^\bullet)$. Then the cohomology spaces of these complexes are related by the functor $\mathrm{Hom}(M, -)$:
$$H^\bullet(\mathrm{Hom}(M, C^\bullet)) = \mathrm{Hom}(M, H^\bullet(C^\bullet)).$$

Exercise 1.1.3. (Requires Category Theory) Verify that the *forgetful functor* from the category of $\mathbb{Q}$-vector spaces Vect to the category of abelian groups Ab is fully faithful and describe its essential image.

Problem 1.1.4. Verify that $\mathrm{Hom}(-, \mathbb{Z})$ is not exact by applying it to the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Problem 1.1.5. Show that $\mathrm{Hom}(-, \mathbb{Q})$ is exact.

Exercise 1.1.6. In class, we defined the group operation on Ext (*Baer's sum*). Rewrite the definition explicitly, without using functoriality of Ext. Verify that it indeed gives an abelian group structure on Ext, whose zero and inversion are as described in class.

Challenge 1.1.7. Describe the following functors on the category of abelian groups: $\mathrm{Ext}(\mathbb{Q}, -)$, $\mathrm{Ext}(-, \mathbb{Z}/p\mathbb{Z})$ (where p is a prime), $\mathrm{Ext}(-, \mathbb{Z})$. (The goal here is to make the answer as explicit as possible, but it is hard to get anywhere. If you get a simple answer, you probably made a mistake. This challenge is more about understanding why these functors are hard.)

The next four problems refer to properties of the Ext functor that are parallel to the properties of Hom. If you have not seen the corresponding properties of the Hom functor, you should verify them as an exercise.

Problem 1.1.8. Let M_1, M_2, N be abelian groups. Construct a natural isomomorphism

$$\mathrm{Ext}(M_1 \oplus M_2, N) = \mathrm{Ext}(M_1, N) \oplus \mathrm{Ext}(M_2, N).$$

(Hint: it helps to think about the natural maps $M_{1,2} \to M_1 \oplus M_2$ and $M_1 \oplus M_2 \to M_{1,2}$.)

Problem 1.1.9. Dually, let M, N_1, N_2 be abelian groups. Construct a natural isomomorphism

$$\mathrm{Ext}(M, N_1 \oplus N_2) = \mathrm{Ext}(M, N_1) \oplus \mathrm{Ext}(M, N_2).$$

Problem 1.1.10. More generally, let $N \in \mathrm{Ab}$, and let M_α be a family of abelian groups, not necessarily finite. Construct a natural isomorphism

$$\mathrm{Ext}(\bigoplus M_\alpha, N) = \prod \mathrm{Ext}(M_\alpha, N).$$

Note the direct sum on the left and the product on the right; since the family is not assumed to be finite, these are different operations!

Remark: If you know the definition of direct/inverse limits (a.k.a. colimits/limits, injective/projective limits), you may try generalizing this problem using (filtered) direct and inverse limits... but it probably won't work! Can you find a counterexample? Do you see why it fails?

Problem 1.1.11. Dually, let $M \in \mathrm{Ab}$, and let N_α be a family of abelian groups, not necessarily finite. Construct a natural isomorphism

$$\mathrm{Ext}(M, \prod N_\alpha) = \prod \mathrm{Ext}(M, N_\alpha).$$

(Now its product on both sides.)

Problem 1.1.12. Suppose $M, N \in \mathrm{Ab}^{fg}$ (recall that this means M and N are finitely generated abelian groups). Prove that $\mathrm{Ext}(M, N)$ is finitely generated as well. (Hint: This is one place where using the classification may be appropriate.)

Exercise 1.1.13. Define the connecting homomorphism between Hom and Ext in the situation 'dual' to the one considered in class. Thus, if M is a module and $0 \to N_1 \to N_2 \to N_3 \to 0$ is an exact sequence, you need to construct the map

$$\mathrm{Hom}(N_1, M) \to \mathrm{Ext}(N_3, M).$$

Exercise 1.1.14. Check that the Hom/Ext long exact sequence (corresponding to an abelian group M and a short exact sequence of abelian group) is exact. Note that the long exact sequence comes in two flavors, so there are two independent checks involved. The most interesting part is to check exactness at the connecting homomorphism between Hom and Ext.

Exercise 1.1.15. Formulate and prove the functoriality property of the connecting homomorphism.

Problem 1.1.16. Verify that for any abelian group M, $\mathrm{Ext}(-, M)$ is right exact. The problem is much more interesting to solve from 'first principles' using only the notion of extensions. But if you must use resolutions (i.e., the approach to Ext by generators and relations) this is fine, too.

Challenge 1.1.17. Verify that for any abelian group M, $\mathrm{Ext}(M, -)$ is right exact. (This is not hard using injective resolutions, but doing this directly in terms of extensions requires thought.)

Exercise 1.1.18. Show that if M and N are finitely generated abelian groups, then the groups $\mathrm{Hom}(M, N)$ and $\mathrm{Ext}(M, N)$ are finitely generated as well.

Exercise 1.1.19. Show that if M and N are abelian groups and the number k is such that either $kM = 0$ or $kN = 0$, then $kA = 0$ for $A = \mathrm{Hom}(M, N)$ and for $A = \mathrm{Ext}(M, N)$.

Problem 1.1.20. An abelian group M is projective if and only if M is free. (Note that M is not assumed to be finitely generated.)

Problem 1.1.21. An abelian group is injective if and only if it is divisible.

Problem 1.1.22. Construct a functorial morphism

$$\mathrm{Ext}(M, \mathbb{Z}) \otimes_{\mathbb{Z}} N \to \mathrm{Ext}(M, N)$$

such that if $N = \mathbb{Z}$, it is the tautological map. Prove that this map is an isomorphism if either M or N are finitely generated. What can go wrong if both M and N are infinitely generated?

1.2. **Homological algebra in the category of modules.**

Exercise 1.2.1. Suppose R is a PID. Show that a module is projective iff it is free, and a module is injective iff it is divisible.

Exercise 1.2.2. Describe projective and injective modules over the matrix ring $Mat_n(k)$, where k is a field.

Problem 1.2.3. Let R be a finite-dimensional algebra over the field k. Consider the dual space

$$\mathrm{Hom}_k(R, k) = R^\vee,$$

and turn it into a left R-module by using the right action of R on itself. Show that the R-module $R^\vee$ is injective, that a direct sum of (possibly infinitely many) copies of $R^\vee$ is injective, and that a module is injective iff it is a direct summand of such a direct sum.

(Side question: is it important that R is finite-dimensional?)

Problem 1.2.4. Verify the following:

(1) The direct sum (not necessarily finite!) of modules is projective iff all summands are projective;
(2) The Cartesian product (not necessarily finite!) of modules is injective iff all factors are injective.

(Side question: is it true that the direct limit of projective modules is projective?)

Problem 1.2.5. Suppose $f : R \to S$ is a morphism of rings.

(1) Show for a projective R-module M, its extension of scalars $S \otimes_R M$ is projective as an S-module.
(2) Suppose in addition that S is flat as a right R-module. Show that an injective S-module is also injective as an R-module.

(Side question: how to generalize this statement to a claim about behaviour of projectivity and injectivity under adjoint functors?)

Problem 1.2.6. Suppose $f : R \to S$ is a morphism of rings.

(1) Let M be an R-module. Consider the abelian group $M' := \mathrm{Hom}_R(S, M)$, where S is viewed as a left R-module in the natural way. Turn it into a left S-module using the right action of S on itself.
(2) Show that for every S-module N, we have an isomorphism

$$\mathrm{Hom}_R(N, M) = \mathrm{Hom}_S(N, M').$$

(3) Prove that if M is injective, then so is M'.

Exercise 1.2.7. Show that for any additive functor $\mathrm{Mod}_R \to \mathrm{Ab}$ and any two modules M and N, the natural homomorphism

$$F(M) \oplus F(N) \to F(M \oplus N)$$

is an isomorphism.

Problem 1.2.8. Let $F : \mathrm{Mod}_R \to \mathrm{Ab}$ be an additive (covariant) functor with the following properties:

(1) F is right exact;

(2) F commutes with arbitrary direct sums: that is, for any collection of modules M_α, the natural map
$$\bigoplus F(M_\alpha) \to F(\bigoplus M_\alpha)$$
is an isomorphism.

Show that there exists a right R-module N and a functorial isomorphism
$$F(-) \simeq N \otimes_R -.$$

Problem 1.2.9. Let $F : \mathrm{Mod}_R \to \mathrm{Ab}$ be an additive contravariant functor with the following properties:

(1) F is left exact;
(2) F sends direct sums to direct products: that is, for any collection of modules M_α, the natural map
$$F(\bigoplus M_\alpha) \to \prod F(M_\alpha)$$
is an isomorphism.

Show that there exists a left R-module N and a functorial isomorphism
$$F(-) \simeq \mathrm{Hom}_R(-, N).$$

Exercise 1.2.10. Let M be a right R-module. Prove that the following properties are equivalent:

(1) M is flat: that is, the functor $M \otimes_R -$ is exact.
(2) $\mathrm{Tor}_1^R(M, -) = 0$.
(3) $\mathrm{Tor}_k^R(M, -) = 0$ for all M.

Formulate and prove similar equivalences for projective and injective modules.

Problem 1.2.11. Show that M is flat if an only if $\mathrm{Tor}_1(M, N) = 0$ for any *finitely generated* R-module N. (In fact, it suffices to assume that N is *cyclic*, that is, generated by a single element.) (This problem may be harder or easier depending on what properties of tensor product you are familiar with.)

Challenge 1.2.12. Suppose $R = \mathbb{Z}$, so we work with $\mathrm{Ab} = \mathrm{Mod}_R$. Is it true that $M \in \mathrm{Ab}$ is projective if and only if $\mathrm{Ext}^1(M, N) = 0$ for any finitely generated abelian group N?

Exercise 1.2.13. A module M over a PID is flat if and only if M is torsion-free.

Exercise 1.2.14. Let $R = \mathbb{Z}/4\mathbb{Z}$, and consider $M = \mathbb{Z}/2\mathbb{Z}$ as a R-module.

(1) Write a projective resolution of M. (Can you write an injective resolution, while you are at it?)
(2) Compute $\mathrm{Tor}_i^R(M, M)$.
(3) Compute $\mathrm{Ext}_R^i(M, M)$.

Problem 1.2.15. Suppose $P_\bullet \to M$ is a resolution of an R-module M. Recall that the complex $(P_\bullet \to M)$ (together with the augmentation map $P_0 \to M$) is acyclic. Show that it is null-homotopic iff M is projective. (By definition, a complex is null-homotopic if its identity map is homotopic to the zero map; this implies acyclicity.) (Comment: it may be worth it to look for an elegant solution.)

Exercise 1.2.16. (1) Let $P_\bullet$ be a bounded above complex of projective R-modules. Show that it is null-homotopic iff it is acyclic.

(2) Let $R = \mathbb{Z}/4\mathbb{Z}$, and consider the unbounded acyclic complex of projective modules
$$\cdots \to R \to R \to R \to \dots,$$
where each map is the multiplication by 2. Show that it is not null-homotopic (so the boundedness condition in the first part is essential).