When Models became Polish

An introduction to the Topological Vaught Conjecture

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Prologue
The Continuum Hypothesis

The Continuum Hypothesis

Continuum Hypothesis (CH)

Every set of reals $A \subseteq \mathbb{R}$ is either countable or $|A| = |\mathbb{R}|$.

Cantor's approach: prove it for simple sets and work your way towards more complicated sets.

Polish Spaces

Definition

A *Polish space* is a completely metrizable separable topological space.

Examples: $2^\omega, \omega^\omega, [0,1]^\omega, \mathbb{R}^n, \ldots$

Definition

A Polish space X is *perfect* if it contains no isolated points.

Theorem

The Cantor space 2^ω embeds into any nonempty perfect Polish space.

The Perfect Set Property

Definition

A Polish space X has the *perfect set property* if it is either countable or contains a perfect subset. In particular, X is not a counterexample of the CH.

Theorem (Cantor-Bendixon)

Every Polish space can be written uniquely as $P \cup C$, where P is perfect and C is countable.

In particular, G_δ and F_σ subsets of a Polish space are not satisfy CH.

More complicated sets...

Theorem (Hausdorff, Alexandrov 1916)

Every Borel subset of a Polish space has the perfect set property.

Definition

A subset A of a Polish space X is *analytic (or $\mathbf{\Sigma}^1_1$)* if there is a Polish space Y and a continuous function $f : Y \to X$ such that $f[X] = A$.

Theorem (Suslin 1917)

Every analytic subset of a Polish space has the perfect set property.

Part 0
Polish Groups

Polish Groups

A group G endowed with a topology that makes the map

$$(x, y) \mapsto xy^{-1}$$

continuous is a topological group. If the topology is Polish, we call G a *Polish group*.

Example

S_∞, the symmetric group on ω, with the topology inherited from ω^ω is a Polish group.

Moreover, the closed subgroups of S_∞ are exactly the automorphism groups of countably infinite structures in a countable relational language.

Polish G-spaces

If G is a Polish group, X is a Polish space, and $a : G \times X \to X$ is a continuous group action, we say that X is a *Polish G-space*.

Example

Let $\mathcal{L} = \{R_i\}_{i \in I}$ be a countable relational language where R_i is n_i-ary. Then,

$$Mod_{\mathcal{L}} = \prod_{i \in I} 2^{\omega^{n_i}}$$

is the space of countably infinite structures in the language $\mathcal{L}$ (each $x \in Mod_{\mathcal{L}}$ is the atomic diagram of a structure $\mathcal{A}_x$ with universe ω).

The *logic action* $J_{\mathcal{L}}$ of S_∞ on $Mod_{\mathcal{L}}$ is defined by

$$J_{\mathcal{L}}(g, x) = y \quad \text{if and only if} \quad \mathcal{A}_x \cong \mathcal{A}_y$$

$J_{\mathcal{L}}$ is continuous, which makes $Mod_{\mathcal{L}}$ a Polish S_∞-space.

Silver & Burgess

Theorem (Silver 1980)

If E is a $\mathbf{\Pi}^1_1$ equivalence relation on a Polish space X, then X/E has the perfect set property.

Theorem (Burgess 1978)

If E is a $\mathbf{\Sigma}^1_1$ equivalence relation on a Polish space X, then $|X/E| \leq \aleph_1$ or $|X/E| = 2^{\aleph_0}$

Part 1
Vaught's Conjecture

Denumerable models of complete theories

Vaught's Conjecture (1961)

Any first-order theory in a countable language has either countably-many, or $2^{\aleph_0}$ non-isomorphic countable models.

This is trivially true if we assume CH.

Denumerable models of complete theories

Vaught's Conjecture (1961)

Any first-order theory in a countable language has either countably-many, or $2^{\aleph_0}$ non-isomorphic countable models.

This is trivially true if we assume CH.
For a first-order theory T,

$$M_T = \{x \in Mod_\tau \mid \mathcal{A}_x \models T\}$$

We say that T has perfectly-many models if M_T has perfectly-many orbits.

Vaught's Conjecture 2.0

Any first-order theory in a countable language has either countably-many, or perfectly-many non-isomorphic countable models.

Since M_T is Borel and the equivalence relation induced by the logic action is $\mathbf{\Sigma}^1_1$, Burgess theorem tells us that there are only three options: T has countably-many models, $\aleph_1$ but not perfectly-many, or T has perfectly-many models.

Notice that Version 2.0 is absolute!

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Very soon, it became clear that this question should be investigated using the infinitary language $\mathcal{L}_{\omega_1,\omega}$.

Infinitary Logic $\mathcal{L}_{\kappa,\lambda}$

Let $\mathcal{L}$ be a language and κ, λ two cardinals with $\kappa \geq \lambda$. The formulas in $L_{\kappa,\lambda}$ are constructed in a similar way as first-order formulas, but we allow conjunctions and disjunctions of $< \kappa$ formulas and strings of (single) quantifiers of length $< \lambda$.

Example

- $L_{\omega,\omega}$ is the usual first-order language.
- $L_{\omega_1,\omega}$ is the language allowing countably many conjunctions and disjunctions in each formula, but using only finitely-many quantifiers.

Back-and-Forth relations

For a countable structure $\mathcal{A}$, and $\bar{a}, \bar{b} \in A^n$, we define

- $\bar{a} \equiv_0^n \bar{b}$ if and only if $\mathcal{A} \models \psi(\bar{a}) \leftrightarrow \psi(\bar{a})$ for every quantifier-free formula ψ.
- $\bar{a} \equiv_n^{\alpha+1} \bar{b}$ if and only if for all $c \in A$ there is a $d \in A$ such that $\bar{a}c \equiv_\alpha^{n+1} \bar{b}d$ and for all $d' \in A$ there exists $c' \in A$ such that $\bar{a}c' \equiv_\alpha^{n+1} \bar{b}d'$.

Notice that if $\alpha \geq \beta$, then $\equiv_\alpha^n$ refines $\equiv_\beta^n$. Moreover, there is an $\alpha < \omega_1$ where the relations stabilize. Since A is countable, there is an α that works globally, this is the Scott rank of $\mathcal{A}$, which we denote by $SR(\mathcal{A})$.

Even more, for a fixed $\bar{a}$, we can define its equivalence class under $\equiv_\alpha^n$ with a single $\mathcal{L}_{\omega_1,\omega}$ formula $\varphi_{\bar{a},\alpha}(\bar{x})$.

Scott Isomorphism Theorem

Theorem (Scott, 1965)

For every countable structure $\mathcal{A}$, there is an $\mathcal{L}_{\omega_1,\omega}$ sentence φ such that if $\mathcal{B}$ is a countable structure with $\mathcal{B} \models \varphi$, then $\mathcal{A} \cong \mathcal{B}$.

We call φ the Scott sentence of $\mathcal{A}$. Essentially, we construct φ that says all its models have Scott rank no larger than $SR(\mathcal{A})$ and they satisfy $\varphi_{\Lambda,\alpha}$, where Λ is the empty string and $\alpha = SR(\mathcal{A})$.

Another version of Vaught's Conjecture

Vaught's Conjecture 3.0

A single $\mathcal{L}_{\omega_1,\omega}$ sentence has either countably-many or perfectly-many non-isomorphic countable models.

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Theorem (Morley 1970)

Every complete theory T on a countable language has either countably-many, $\aleph_1$ or perfectly-many non-isomorphic countable models.

Morley's theorem is a particular case of Burgess theorem.

An interesting example

Let $\mathcal{L}_0$ be the language with one binary relational symbol. Then, the class of countable ordinals has $\aleph_1$ isomorphism types, but not perfectly-many.

So, the analog of Vaught's Conjecture for theories in $\mathcal{L}_{\omega_1,\omega}$, and for sentences in $\mathcal{L}_{\omega_1,\omega_1}$ and $\mathcal{L}_{\omega_2,\omega}$ are false. In each of this cases, we can construct a theory of sentence whose models are exactly the countable ordinals.

Part 2
Topological Vaught Conjecture

A shift in perspective

We mentioned that the models of a single sentence $\varphi \in \mathcal{L}_{\omega_1,\omega}$ form a Borel subset of $Mod_{\mathcal{L}}$ invariant under the logic action. The converse is also true:

Theorem (López-Escobar)

If $B \subseteq Mod_{\mathcal{L}}$ is Borel and invariant under the logic action, there is an $\mathcal{L}_{\omega_1,\omega}$ sentence such that $B = \{x \in Mod_{\mathcal{L}} \mid \mathcal{A}_x \models \varphi\}$.

This suggests a different phrasing for Version 3.0 of Vaughts Conjecture:

Vaught's Conjecture Version 3.1

Let $\mathcal{L}$ be a countable language. For any $B \subseteq Mod_{\mathcal{L}}$ Borel and invariant under the logic action, the set of orbits of B has the perfect set property.

Infinitely-many new conjectures!

Let G be any Polish group. A Borel G-space X is a standard Borel space with an action that is Borel-measurable.

Topological Vaught's Conjectures (D.E. Miller 1980)

- **TVC1(G):** For any Polish G-space X, X has countably-many or perfectly-many orbits.
- **TVC2(G):** For any Polish G-space X, and any Borel invariant set $B \subset X$, B has countably-many or perfectly-many orbits.
- **TVC3(G):** For any Borel G-space X, X has countably-many or perfectly-many orbits.

$$\textbf{TVC3(G)} \implies \textbf{TVC2(G)} \implies \textbf{TVC1(G)}$$

But not really...

Theorem

For any Polish group G,

$$\textbf{TVC1(G)} \implies \textbf{TVC3(G)}$$

But not really...

Theorem

For any Polish group G,

$$\textbf{TVC1(G)} \implies \textbf{TVC3(G)}$$

One final restatement

Topological Vaught Conjecture

Let $B \subset X$ be a Borel subset of a Polish space, and let $G : G \times B \to B$ a Borel-measurable action. The action has either countably-many or perfectly-many orbits.

Is that even my final form?

- If we change Borel for analytic, the conjecture is false. (order-types of ordered abelian groups)
- If we change Borel for coanalytic, the conjecture is false. (countable ordinals)
- If we change Borel-measurable for σ-algebra containing analytic sets, we have a counterexample.
- If we change Polish for any natural extension, there are counterexamples.

Yes, it is.

Theorem

The Topological Vaught Conjecture is equivalent to Vaught's Conjecture Version 3.0

The End