Graduate Logic Seminar-Taeyoung Em

Assume Gale-Stewart thrm holds. We will prove A.C.

Given a non-empty set Y, we want to find a choice function on $\mathcal{P}(Y) \setminus \{\varnothing\}$.
First pick $y \in Y$.
We let $A = \mathcal{P}(Y)$ and we let our pruned tree $T \subset A^{<\omega}$ be defined as follows:
the first nodes of T, $T(0)$, is $T(0) \coloneqq \{S \subset Y : S \neq \varnothing\}$, i.e. the first play of player I is by choosing some non-empty subset of Y. Given a first node of T, say a non-empty $S \subset Y$, the second nodes of T following this first node are $\{a\}$ for $a \in S$.

From then on, the only nodes of T are $\{y\}$. In other words, if players I and II play within this pruned tree, player I first plays by choosing some non-empty subset $S \subset Y$, then player II plays his first play by choosing some $\{a\}$ with $\{a\} \subset S$, then both players keep choosing $\{y\}$, since there are no other nodes of T other than $\{y\}$ after player II's first play.

Note that any strategy of II induces a choice function on $\mathcal{P}(Y) \setminus \{\varnothing\}$.
We consider the game $G(T, \varnothing)$. By Gale-Stewart, this game is determined.
We claim that player I cannot have a winning strategy. For this, it suffices to show that any non-empty pruned subtree $T' \subset T$ is such that $[T'] \neq \varnothing$, since any strategy σ of player I is a non-empty pruned subtree of T, and hence we cannot have $[\sigma] \subset \varnothing$ if we show that any pruned subtree has some infinite branch.
Let $T' \subset T$ be a non-empty pruned subtree. Pick Z_0 among the first nodes of T'. Since T' is a pruned tree, we may pick Z_1 among the second nodes of T' that follows after Z_0. Then $(Z_0, Z_1, \{y\}, \{y\}, ...) \in [T']$. Hence, this shows that player I cannot have a winning strategy.

By Gale Stewart, player II has a winning strategy in this game, and in particular, has a strategy.