Choice problems in the eW degrees

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Introduction: What

Problems and Weihrauch reduction

Weihrauch reduction is a way of comparing the computational strength of various "problems", represented as partial multifunctions on $\mathbb{N}^{\mathbb{N}}$.

We may think of Weihrauch reduction $f \leq_W g$ as a computation of values of f, given the ability to query g as an oracle *exactly once*.

Formally, we have this reduction if there are computable functionals (Φ, Ψ) such that

1. $\alpha \in \operatorname{dom} f \Rightarrow \Phi(\alpha) \in \operatorname{dom} g$
2. for any $\alpha \in \operatorname{dom} f$ and $\beta \in g(\Phi(\alpha))$, we have $\Psi(\alpha, \beta) \in f(\alpha)$.

The algebras $\mathbb{P}$ & $\mathbb{P}_\sharp$

Define $\mathbb{P}$ to be $\mathcal{P}(\mathbb{N})$ equipped with a binary operation given by

$$AB = \{n : \exists m(\langle n, m\rangle \in A \wedge D_m \subseteq B)\}.$$

The algebra $\mathbb{P}_\sharp$ is the substructure of $\mathbb{P}$ consisting of the c.e. sets. The elements of $\mathbb{P}_\sharp$ are alo called *enumeration operators*.

Dana Scott proved in [Scott, 1976] that both these algebras can interpret the untyped lambda calculus or Schönfinkel/Curry's combinator calculus.

eW-problems and eW-reductions

An **e**W-problem is a partial multifunction from $\mathbb{P}$ to itself. Given problems f, g, we say that $f \leqslant_{\mathbf{e}W} g$ if there are enumeration operators Γ, Δ such that

1. if $A \in \operatorname{dom} f$ then $\Gamma A \in \operatorname{dom} g$,
2. and for any $A \in \operatorname{dom} f$ and $X \in g(\Gamma A)$, $\Delta(A, X) \in f(A)$.

In other words, **e**W-reduction is just Weihrauch reduction where the problems operate on $\mathbb{P}$, and enumeration reduction (i.e. the action of elements in $\mathbb{P}_\sharp$) is our notion of computation.

Introduction: Why

Why formulate the notion of eW-reduction?

First, enumeration operators have a robust computational structure, and their use to study problems-as-multifunctions is intrinsically interesting.

Moreover, it's a notion of computation that works on *positive* information, potentially making some different distinctions between common problems.

Why formulate the notion of eW-reduction?

Second, they are related to an under-studied *realizability topos*.

- A topos is a category theoretic model of a kind of intuitionistic set theory. Realizability toposes are built from a model of computation (see [van Oosten, 2008] for an overview of the area).
- There is a realizability topos where the underlying model of computation is enumeration reduction—the topos $\mathsf{RT}(\mathbb{P}, \mathbb{P}_\sharp)$.
- There is a strong relationship between $\mathcal{D}_{\mathbf{e}W}$ and subtoposes of $\mathsf{RT}(\mathbb{P}, \mathbb{P}_\sharp)$ (see [Kihara, 2023]).

Basic results about eW-reduction

The eW degrees extend the Weihrauch

Proposition. There is an embedding of the Weihrauch degrees into the **e**W degrees.

Proof sketch.

- Using the injective function $\mathsf{gr} : \mathbb{N}^\mathbb{N} \to \mathcal{P}(\mathbb{N})$, replace a Weihrauch problem f with $\tilde{f}$ so that $\mathsf{gr}(\alpha) \in \tilde{f}(\mathsf{gr}(\beta))$ iff $\alpha \in f(\beta)$.

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- We can replace each Turing functional in a reduction $f \leq_W g$ with enumeration operators that witness $\tilde{f} \leqslant_{\mathbf{e}W} \tilde{g}$. (Think about the graph of the computable function $\omega^{<\omega} \to \omega^{<\omega}$ that defines the functional.)

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- We can replace each Turing functional in a reduction $f \leq_W g$ with enumeration operators that witness $\tilde{f} \leqslant_{\mathbf{e}W} \tilde{g}$. (Think about the graph of the computable function $\omega^{<\omega} \to \omega^{<\omega}$ that defines the functional.)
- Now we want $\tilde{f} \leqslant_{\mathbf{e}W} \tilde{g}$ to imply $f \leq_W g$. Given an enumeration operator Γ, we may pick a computable enumeration γ of Γ and define a functional Φ such that $\Phi(\alpha)(n)$ is found by searching longer and longer portions of γ and α to find when $\Gamma(\mathrm{gr}(\alpha))$ outputs a pair $\langle n, k\rangle$.

The eW degrees extend the Weihrauch

This mapping is not surjective.

Let $g :\subseteq \mathbb{P} \rightrightarrows \mathbb{P}$ have domain consisting of a single 1-generic G, to which every element of $\mathbb{P}$ is a solution. Suppose that there is a Weihrauch problem f with $g \equiv_{\mathbf{e}W} \tilde{f}$.

Since G is quasi-minimal, and every element in the domain of $\tilde{f}$ is total, $\Gamma : \operatorname{dom} g \to \operatorname{dom} \tilde{f}$ occurring in a reduction $g \leqslant_{\mathbf{e}W} \tilde{f}$ must send G to a computable element.

But now a reduction $\tilde{f} \leqslant_{\mathbf{e}W} g$ must send that computable element of $\operatorname{dom} \tilde{f}$ to G, requiring G to be a c.e. set, contradicting 1-genericity.

The problem *id*

Definition. The problem *id* is the identity function on $\mathbb{P}$.

Proposition.

1. $f \leqslant_{\mathbf{e}W} id$ if and only if there is an enumeration operator Γ such that for all $A \in \operatorname{dom} f$, $\Gamma A \in f(A)$.
2. $id \leqslant_{\mathbf{e}W} f$ if and only if f has a c.e. instance.

Proof.

1. Let the reduction be witnessed by (Γ, Δ); then $\Delta(A, \Gamma A) \in f(A)$ and can be coded by a single enumeration operator.
2. $\varnothing$ is an *id*-instance, so if (Γ, Δ) witness a reduction, $\Gamma\varnothing$ must be a (c.e.) f-instance.

Closed choice problems

Definition. A computable metric space X is a separable metric space with a listing of a dense set $(p_n)_{n\in\omega}$ such that the distance function $(n, m) \to d(p_n, p_m)$ is computable.

The *closed choice problem* on a complete metric space X, $\mathsf{C}_X :\subseteq \mathbb{P} \rightrightarrows \mathbb{P}$, takes an encoding of an open set with non-empty complement, and returns (the name of) an element of that complement.

Closed choice problems

In the Weihrauch setting, open complements of closed sets are coded by enumerations of open balls—i.e. of pairs (n, r) representing an open ball of radius r around p_n.

In the $\mathbf{e}W$ setting, we may simply take the *set* of open balls instead of a listing of the open balls. In the case of Baire or Cantor space, we may equivalently represent open sets by sets of finite strings, and for $\mathbb{N}$ we represent open sets by themselves.

A first example: $\mathsf{C}_{\mathbb{P}}$

A natural topology on $\mathbb{P}$ is the *positive information topology*: the basic open sets are of the form $O_a := \{A \in \mathcal{P}(\mathbb{N}) \,:\, a \subseteq A\}$ for a finite. (Note: this isn't actually a metric space.)

We may represent open sets O by a set $I \subseteq \mathbb{N}$ such that $O = \bigcup_{i \in I} O_{D_i}$.

Proposition. $\mathsf{C}_{\mathbb{P}} \equiv_{\mathbf{e}W} id$.

Proof. Every closed set of $\mathbb{P}$ contains $\varnothing$, so the enumeration operator coding $\lambda x.\varnothing$ computes solutions from instances.

C_{-} and $\widetilde{\mathsf{C}_{-}}$

The $\mathbf{e}W$ versions of choice problems tend to fall strictly above their Weihrauch counterparts.

For instance, $\widetilde{\mathsf{C}_{\mathbb{N}}} <_{\mathbf{e}W} \mathsf{C}_{\mathbb{N}}$, the reduction being easy—one need only take the graph of a function to its range, and solutions in both cases are graphs of paths.

On the other hand, $\varnothing$ is an instance of $\mathsf{C}_{\mathbb{N}}$, so consider two distinct singleton closed sets A, B. Since $\varnothing \subseteq A, B$, we must have $\Gamma\varnothing \subseteq \Gamma A, \Gamma B$ for any enumeration operator Γ; but the image of Γ consists of graphs of total functions, so it has to be constant.

The arguments for $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ and $\mathsf{C}_{2^{\mathbb{N}}}$ are similar.

$\mathsf{C}_{\mathbb{N}}$ & $\mathsf{UC}_{\mathbb{N}}$

The first interesting separation that develops in the $\mathbf{e}W$ setting concerns $\mathsf{C}_{\mathbb{N}}$ and its restriction to singletons, $\mathsf{UC}_{\mathbb{N}}$.

Fact. $\widetilde{\mathsf{C}_{\mathbb{N}}} \equiv_{\mathbf{e}W} \widetilde{\mathsf{UC}_{\mathbb{N}}}$.

Proposition. $\mathsf{UC}_{\mathbb{N}} <_{\mathbf{e}W} \mathsf{C}_{\mathbb{N}}$

$\mathsf{C}_{\mathbb{N}}$ & $\mathsf{UC}_{\mathbb{N}}$

Proof. The reduction is immediate from the fact that unique choice is just a restriction of closed choice.

For strictness, suppose we had a reduction $\mathsf{C}_{\mathbb{N}} \leqslant_{\mathbf{e}W} \mathsf{UC}_{\mathbb{N}}$ witnessed by (Γ, Δ). Then $\Gamma\varnothing$ is the complement of a singleton, and for any other $A \in \operatorname{dom} \mathsf{C}_{\mathbb{N}}$, ΓA must both be the complement of a singleton, and a superset of $\Gamma\varnothing$. The only way this can happen is if $\Gamma A = \Gamma\varnothing$, meaning Γ must be constant.

Now consider $\{k\} = \Delta(\varnothing \oplus \{n\})$, where $n \in \overline{\Gamma A}$; then $\Delta(\{k\} \oplus \{n\})$ must also contain k by monotonicity, and cannot be outputting any subset of $\overline{\{k\}}$.

Commentary

In general, we think of instances and solutions as codes for elements of mathematical objects (e.g. points in spaces, or closed sets of topologies).

Here we see a substantial difference in behavior depending on *what* information our codes contain—positive and negative information, or just positive.

I don't know what this means, but it's pretty cool.

$\mathsf{C}_{2^{\mathbb{N}}}$ **and** WKL

We have $\widetilde{\mathsf{C}_{2^{\mathbb{N}}}} \equiv_{\mathbf{e}W} \widetilde{\mathsf{WKL}}$ as a standard result from the Weihrauch degrees. With positive and negative information, these are two different representations of the same thing.

On the other hand,

Proposition. $\mathsf{C}_{2^{\mathbb{N}}} \mid_{\mathbf{e}W} \mathsf{WKL}$

Proof. Suppose in each case below that (Γ, Δ), towards a contradiction, witnesses the specified reduction.

1. ($\mathsf{C}_{2^{\mathbb{N}}} \not\leq_{\mathbf{e}W}$ WKL). Consider the set $\Gamma\varnothing$ ($\varnothing$ coding the full closed set of Cantor space). Then $\Gamma\varnothing$ is an infinite c.e. tree, such that for any $\mathsf{C}_{2^{\mathbb{N}}}$-instance C we have $\Gamma\varnothing \subset \Gamma C$. So there is a $\mathbf{0}''$-computable path P such that $\Delta(C \oplus P) \in \mathsf{C}_{2^{\mathbb{N}}}(C)$ for any $C \in \mathrm{dom}\mathsf{C}_{2^{\mathbb{N}}}$.

   Now let C be the complement of a $\mathbf{0}''$-computable tree with no $\mathbf{0}''$-computable paths. Then $\Delta(C \oplus P) \leq_e C \oplus P \leq_T \mathbf{0}''$. Since $\Delta(C \oplus P)$ is a total object (elements of Cantor space are total functions), this makes it $\mathbf{0}''$-computable, which is impossible if (Γ, Δ) is a reduction.

$\mathsf{C}_{2^{\mathbb{N}}}$ **and** WKL

Proof. Suppose in each case below that (Γ, Δ), towards a contradiction, witnesses the specified reduction.

2. (WKL $\not\leq_{\mathbf{e}W}$ $\mathsf{C}_{2^{\mathbb{N}}}$). Consider the full tree $T = 2^{<\omega}$, and the closed set ΓT. Again, ΓT, as a set of strings coding the complement of a closed set, is c.e., so the closed set in question is Π^0_1; moreover, it's a subset of every other closed set in the image of Γ.

   So we fix a Δ^0_2 element $P \in \Gamma T$. Now choose a Δ^0_2 tree T with no Δ^0_2 paths, and proceed as above.

In fact, for very simlar reasons, we even have WKL $|_{\mathbf{e}W}$ $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$!

Commentary

We've now seen that we can see that an equivalence from the Weihrauch setting may break down in both directions, or only one.

The trend in these proofs of exploiting $\subseteq$-monotonicity of enumeration operators is common to many proofs in the setting.

Some generalities about *WKL* & closed choice

WKL **can't reduce to any closed choice problem.** We can generalize the previous instances. This plays on the fact that the set of all trees has a largest element under inclusion, and we can make trees of any complexity with even more complex paths.

No problem to which every instance has a computable solution can compute WKL. The existence of computable trees T with no computable path means that any $\Delta(T, A)$, for A computable, must fail to be a path in such a T.

Unique choice problems are always weaker. There's nothing special about $\mathbb{N}$ in our analysis of $C_{\mathbb{N}}$ and $UC_{\mathbb{N}}$.

The many questions remaining

Questions about the $W/\mathbf{e}W$ relationship

Are the Weihrauch and $\mathbf{e}W$ degrees non-isomorphic? Is there a first order difference between them?

In the Weihrauch degrees *id* is definable as the greatest strong minimal cover, and the degrees below *id* are isomorphic to the Medvedev degrees.

The $\mathbf{e}W$ degrees below the identity are isomorphic to the Dyment degrees. If *id* is also the greatest strong minimal cover here, we have a first order difference; but it's not clear that a similar proof works.

Questions coming from the topos

The typical representations of certain spaces in the style of the (enumeration) Weihrauch degrees do not necessarily correspond to the versions of these spaces *internal* to $\mathsf{RT}(\mathbb{P}, \mathbb{P}_\sharp)$.

Do the proper internal versions have a natural mathematical meaning? How strong are the problems involving them?

Questions coming from the topos

Weihrauch problems typically come from maps between *represented spaces*, which are pairs (X, δ_X) where $\delta_X : \mathcal{A} \to X$ is a partial surjection from some sort of computational space (e.g. Baire space or $\mathbb{P}$) that we think of as providing "names" for elements of X.

We obtain a "name version" of a problem $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ by just considering $\delta_Y^{-1} \circ f \circ \delta_X$.

We can do this because elements of the same represented space have disjoint sets of names.

Questions coming from the topos

Instead of assigning names via partial surjections, we could consider $(X, \Vdash_X)$ where $\Vdash_X \subseteq \mathcal{A} \times X$ is a surjective *relation*. (An object called an *assembly* in realizability theory.)

The right notion of the "name version" of a problem changes: we must consider both the name and the named elements together, giving us problems of the form $f \subseteq \mathcal{A} \times X \rightrightarrows \mathcal{A}$.

Problems of this form, and the appropriate notions of reduction, were first studied in detail by [Bauer, 2022] and extended in [Kihara, 2023] and [Kihara, 2022] (arXiv preprint).

Questions coming from the topos

As oracles, we may think of an extended problem $f :\subseteq \mathcal{A} \times X \rightrightarrows \mathcal{A}$ as computations in which we may query f about some element in $\mathcal{A}$, but in which an omniscient advisor chooses a secret input from X to improve our chances of computing correctly.

What does the landscape of the eW problems look like within this extended setting?

Questions coming from the topos

An example class of problems coming from $\mathrm{RT}(\mathbb{P}, \mathbb{P}_\sharp)$ is the following. Let $p \subseteq \mathbb{P}$, and define the problem $p^{\Rightarrow}$ as follows:

- Domain: $\{(A \,|\, q) \in \mathbb{P} \times \mathcal{P}(\mathbb{P}) \;:\; (\forall B \in p)(AB \in q)\}$
- Solutions: $p^{\Rightarrow}(A|q) = q$

This corresponds to forcing the truth value represented by p in $\mathrm{RT}(\mathbb{P}, \mathbb{P}_\sharp)$ to be true. As an oracle, it corresponds to computations in which we have access to a "phantom element of p."

Where do various natural problems fall in relation to those of the form $p^{\Rightarrow}$?

Hybrid questions

We may even use this expanded notion of problem to add new refinements of known problems. Here's a nonce example, $\mathsf{WKL}_{/}$

- $\operatorname{dom} \mathsf{WKL}_{/}$ is the set of pairs $(T \mid \sigma)$ where T is an infinite tree and σ is an initial segment of a path through T.
- $\mathsf{WKL}_{/}(T \mid \sigma)$ is a path in T with initial segment σ.

As an oracle, this corresponds to being able to query WKL, with the aid of an omniscient advisor who can prune all of the tree except for the branch starting with σ.

Are there problems that turn out to be "more related" considering these extended problems? (E.g. are there problems incomparable to WKL that are comparable to $\mathsf{WKL}_{/}$?)

Thanks!

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