Bun_r^d's Cohomology and Atiyah-Bott Classes

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August 6, 2025

Definitions

Definition

Recall that $\mathrm{Bun}_G X$ denotes the algebraic stack of principle G-bundles on an algebraic curve X of genus g. We work over an algebraically closed field of characteristic zero.

Recall that one can consider the category of Λ-modules on a topos T and that Mod_Λ is an abelian category with enough injectives.
So, one can consider sheaf cohomology on stacks.

Definition

A sheaf F on an algebraic stack F is the datum $\{F_{U,u}\}$ where $u : U \to M$ are morphisms from $U \to M$ a scheme U and together with maps $\phi_{UV} : f^{1} F_{U,u} \to F_{V,v}$ s.t. $u \circ f = v$.

Cohomology II

- Čech Cohomology: $H^*(M, F)$ can be computed from an atlast $u : U \to M$ via a spectral sequence
$$E_1^{p,q} = H^q(U_p := \times^p U, F_{U_p}) \Rightarrow H^{p+q}(M, F).$$
- Künneth Formula. The cohomology ring $H^*(M \times N, p^*F \otimes q^*G)$ is the tensor product $H^*(M, p^*F) \otimes H^*(N, q^*G)$.
- Gysin sequence exists. And in particular there are isomorphisms $H^r(X, F) \xrightarrow{\sim} H^r(U, F)$ whenever $U \subseteq X$ is an open substack whose complement has codimension c and $0 \leq r < 2c - 1$.

Cohomology of BGL_n

Theorem

$H^*(BGL_n, \mathbb{Q}_\ell) = \mathbb{Q}_\ell[c_1, \dots, c_n]$ *and* $\deg c_i = 2i$.

Proof.

Consider the diagram (top horizontal arrow is a G_m-torsor)
$$\begin{array}{ccc} \mathbb{A}^n - \{0\} & \longrightarrow & \mathbb{P}^n \\ \downarrow & & \downarrow p \\ * & \longrightarrow & BG_m \end{array}.$$

Now p has fibres $\mathbb{A}^n - \{0\}$. So, $R^i p_* \mathbb{Q}_\ell = \begin{cases} \mathbb{Q}_\ell & i = 0 \\ 0 & i < 2n - 1 \end{cases}$. By Leray's spectral sequence, $H^*(BG_m, \mathbb{Q}_\ell) = H^*(\mathbb{P}^{n-1}, \mathbb{Q}_\ell)$ for $* < 2n - 1$. Now letting $n \to \infty$, one deduces that $H^*(BG_m, \mathbb{Q}_\ell) = \mathbb{Q}_\ell[c_1]$ like claimed.
For GL_n for general n, one relplaces $\mathbb{A}^N - \{0\}$ with $M(n, N)_{rk=n}$ (i.e. $n \times N$-matrices of rank n). □

Chern Classes

Here, we review some formulas regarding Chern classes. First off, if L is a line bundle, then $c_i(L^{\oplus d})$ can be computed:

$$c_i(L^{\oplus d}) = \sigma_i(L, L, \ldots, L)$$

where σ_i are elementary symmetric polynomials. So for example,

$$c_1(L^{\oplus d}) = dc_1(L).$$

Or

$$c_2(L^{\oplus 2}) = c_1(L)c_1(L) \qquad \& \qquad c_2(L^{\oplus 3}) = 3c_1(L)^2.$$

In general, there is a recursive definition for the elementary symmetric polynomials in n variables

$$\sigma_1(X_1, \ldots, X_n) = \sum_{j=1}^{n} X_j \qquad \& \qquad \sigma_k(X_1, \ldots, X_n) = \sum_{j=1}^{n-k+1} X_j \sigma_{k-1}(X_{j+1}, \ldots, X_n)$$

Atiyah-Bott Classes

There is a universal family of vector bundles $\mathcal{E}_{univ}$ over $X \times \mathrm{Bun}_n^d$. From there, we get a map

$$c_{univ} : X \times \mathrm{Bun}_n^d \to BGL_n$$

by Yoneda's Lemma. Then, there's a map

$$univ^* : H^*(BGL_n, \mathbb{Q}_\ell) \to H^*(X \times \mathrm{Bun}_n^d, \mathbb{Q}_\ell) \overset{\text{Künneth}}{=} H^*(X, \mathbb{Q}_\ell) \otimes H^*(\mathrm{Bun}_n^d, \mathbb{Q}_\ell).$$

Assume $H^*(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \cdot 1 \oplus \bigoplus_{i=1}^{2g} \mathbb{Q}_\ell \cdot \gamma_i \oplus \mathbb{Q}_\ell \cdot [pt]$. Then the we can write

$$c_i(\mathcal{E}_{univ}) := univ^*(c_i) = 1 \otimes a_i + \sum_{j=1}^{2g} \gamma_j \otimes b_i^j + [pt] \otimes f_i$$

where $\deg a_i = 2i$, $\deg b_i^j = 2i - 1$ and $\deg f_i = 2i - 2$.
The cohomology classes of a_i, b_i^j, f_i are called **Atiyah-Bott classes**.

Atiyah-Bott Formula

Theorem

The cohomology ring of $\mathrm{Bun_n^d}$ *is given by*

$$H^*(\mathrm{Bun}_n^d, \bar{\mathbb{Q}_\ell}) = \bar{\mathbb{Q}_\ell}[a_1, \dots, a_n] \otimes \bigwedge_{j=1,\dots,2g, i=1,\dots,n} [b_j^i] \otimes \bar{\mathbb{Q}_\ell}[f_1, f_2, \dots, f_n].$$

Example

$H^*(\mathrm{Bun}_1^d, \bar{\mathbb{Q}_\ell}) = H^*(\mathbb{G}_m, \bar{\mathbb{Q}_\ell}) \otimes H^*(\mathrm{Pic}^d(X), \bar{\mathbb{Q}_\ell}) = \bar{\mathbb{Q}_\ell}[a] \otimes H^*(\mathrm{Pic}^d(X), \bar{\mathbb{Q}_\ell})$.
This formula follows from the Künneth formula and $\mathrm{Bun}_1^d \cong B\mathbb{G}_m \times \mathrm{Pic}^d(X)$.

Assume $n > 1$. There are two key steps and I provide some general remarks.

- Show that the (a_i, b_j^i, f_i) include into the cohomology ring and freely generate some subalgebra. This is not as difficult to do.
- Show that this subalgebra is actually everything. This is quite difficult to do and has intricacies that I cannot discuss here.

Step 1. Let $\underline{d} = (d_1, \ldots, d_n)$ denote a partition of d into n parts. Then define $\oplus_{\underline{d}} : \prod \mathrm{Bun}_1^{d_i} \to \mathrm{Bun}_n^d$ given by $(\mathcal{L}_i) \to (\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n)$. Then, using the formula for Chern classes of direct sums, and setting $c_1 \mathcal{L}_{univ}^{d_i}) = 1 \otimes A_i + \sum_{j=1}^{2g} \gamma_j \otimes B_i^j + [pt] \otimes d_i$, we get

$$\oplus_{\underline{d}}^*(c_i(\mathcal{E}_{univ})) = \sigma_i(c_1(\mathcal{L}_{univ}^{d_1}, \ldots, c_1(\mathcal{L}_{univ}^{d_n})))$$

This formula equals

$$\sigma_i(A_1, \ldots, A_n) + \sum_{j,k} \gamma_j \otimes \partial_k \sigma_i(A_1, \ldots, A_n) B_k^j +$$
$$\sum_{j+m=2g+1,k,l} [pt] \otimes \partial_k \partial_l \sigma(A_1, \ldots, A_n) B_k^j B_l^m + \sum_{j,k} [pt] \otimes \partial_k \sigma_i(A_1, \ldots, A_n) d_k$$

One can use the formula from the previous page.

$$\begin{array}{ccc}
H^*(\mathrm{Bun}_n^d) & \longrightarrow & \prod_{\underline{d}} \bar{\mathbb{Q}}_\ell[A_1, \dots, A_n] \otimes \bigwedge[B_i^j] \\
\uparrow & & \uparrow \\
\bar{\mathbb{Q}}_\ell[a_i, f_i] \otimes \bigwedge[b_i^j] & \hookrightarrow & \bar{\mathbb{Q}}_\ell[A_i] \otimes \bigwedge^j[B_i^j] \otimes \bar{\mathbb{Q}}_\ell[D_1, \dots, D_n]/(\sum D_i = d)
\end{array}$$

1) Bottom horizontal arrow is $a_i \to \sigma_i(A_1, \dots, A_n)$, $b_i^j \to \sum_k \partial_k \sigma_i(A_1, \dots, A_n) B_k^j$, and f_i maps to the last summands of the previous page's formula except one replaces d_k by D_k.
2) RHS vertical map sends D_i to $(d_i)_{\underline{d}}$ i.e. $K_1, \dots, K_n$ go to $(d_1, \dots, d_n)$ in one summand, and $(d_1', \dots, d_n')$ in another.
3) The RHS vertical map is injective since we are doing this evaluate map all simulatenously over all partitions of d into n pieces.
4) The bottom horizontal map is injective by construction and linear independence of the $\partial \sigma_i$ (use étaleness of $\mathbb{A}^n \to \mathbb{A}^n / S_n$).
5) Composing along the bottom and then RHS vertical is injective and so that means LHS vertical is injective too!

The last step is trickier and will motivate some of our later discussions on semistable bundles.

Beauville's Trick. For smooth projective schemes X the following is true. Let $\Delta : X \to X \times X$ be the diagonal map. Then the Künneth components of the diagonal $[\Delta] \in H^*(X \times X, \bar{\mathbb{Q}}_\ell)$ generate $H^*(X, \bar{\mathbb{Q}}_\ell)$. (see e.g. `https://arxiv.org/abs/alg-geom/9202024`).

Proof: This fact itself just follows from the projection formula $x = pr_{1*}([\delta] \cap pr_2^* x)$.

Issue: This doesn't work in general for stakcs since Δ need not be an embedding.

Solution: Use stable vector bundles to approximate. (a) Determine the cohomology ring on the moduli of stable bundles which at least as a parameter space given by a scheme. (b) approximate the stack via the moduli of stable bundles. Justify this approximation using parabolic bundles. (c) Increase the codimension by increasing the number of points one fixes for the parabolic structure.

Proof of the Formula V

Theorem

The Künneth components of $[\Delta]$ *generate the cohomolgy ring of* $\mathrm{Bun}_n^{d,stable}$.

Why? According to Beauville's Trick, we only need to compute $[\Delta]$. Consider $E := \mathcal{E}_{univ}$ on $\mathrm{Bun}_n^{d,stable}$ and in particular, $U := RHom(p_{12}^*E, p_{13}^*E)$ on $X \times \mathrm{Bun}_n^{d,stable} \times \mathrm{Bun}_n^{d,stable}$.

Now U can be represented by a two term complex $K^0 \xrightarrow{\delta} K^1$. There are no nontrivial maps between stable bundles of the same rank and degree and so δ has maximal rank outside of $\mathrm{Bun}_n^{d,stable} \times \mathrm{Bun}_n^{d,stable}$.

Thom-Porteous's Formula (Fulton Theorem 14.4) implies $c_{top}(K_0 \to K_1) = [\Delta]$ (one still needs to know something about the codimension of $[\Delta]$).

Grothendieck-Riemann-Roch says

$$ch(U) = pr_{23,*}(pr_X^*(Todd(X)) \cap ch(E)ch(E^\vee)).$$

The RHS is in terms of AB classes. So, $[\Delta]$ is given by AB classes since $c_{top}(K_0 \to K_1)$ appears in $ch(U)$.

Proof of the Formula VI

To upgrade the statement from $\mathrm{Bun}_n^{d,stable}$ to all of Bun_n^d, one can proceed as follows.

Definition

Let $S := \{p_1, \ldots, p_N\} \in X$ be some finite set of closed points.
The stack $\mathrm{Bun}_{n,S}$ of parabolic bundles is given by

$$\mathrm{Bun}_{n,S}^d(T) := \langle (\mathcal{E} \in \mathrm{Bun}_n^d(T), \mathcal{E}_{1,p} \subsetneq \ldots\ldots \mathcal{E}_{n,p} = \mathcal{E}|_{p\times T}), p \in S, \mathcal{E}_{i,p} \text{ is a full flag} \rangle$$

The morphisms in this groupoid are required to preserve the flags. We get a forgetful map $forget : \mathrm{Bun}_{n,S}^d \to \mathrm{Bun}_n^d$.
The fibres of $forget$ are products of flag varieties $\prod_{p\in S} \mathrm{Flag}_n$.

Theorem

By the Leray-Hirsch Theorem,

$$H^*(\mathrm{Bun}_{n,S}^d) = H^*(\mathrm{Bun}_n^d) \otimes \bigotimes_{p\in S} H^*(\mathrm{Flag}_n).$$

Proof of the Formula VII

The proceeding theorem implies $H^*(\mathrm{Bun}_n^d)$ is generated by AB classes iff $H^*(\mathrm{Bun}_{n,S}^d)$ is generated by AB classes and Chern classes given by the flags in $H^*(\mathrm{Flag}_n)$. We call this collection fo classes "canonical classes".

1. One can consider the moduli stack of **parabolically stable bundles**. This depends on a parameter α which is an N-tuple. Chosen well, we can find an open substack $\mathrm{Bun}_{n,S}^{d,\alpha}$ of $\mathrm{Bun}_{n,S}^d$ which has a coarse moduli space $M_{n,S}^d$. and $\mathrm{Bun}_{n,S}^{d,\alpha} \to M_{n,S}^d$ is a G_m-gerbe.
2. Apply Beauville's trick for parabolic bundles to deduce $H^*(\mathrm{Bun}_{n,S}^{d,\alpha})$ is also generated by the AB classes and canonical classes.
3. Letting $N \to \infty$, explain why the codimension for the space of unstable bundles goes to ∞.
4. Justify via the Gysin sequence that

$$H^*(\mathrm{Bun}_n^d, \bar{\mathbb{Q}}_\ell) = \varprojlim_{U \subseteq \mathrm{Bun}_n^d \text{ finite type}} H^*(U, \bar{\mathbb{Q}}_\ell).$$

The limit becomes stationary once the codimension of U is high. Cohomology in degree i agree as soon as codimension is $> i/2$.

Statement of the Generalized Version

Theorem (Heinloth-Schmitt)

Let X be a curve over any field k. Then the cohomology of the connected component $\mathrm{Bun}_G^{\vartheta}$ of Bun_G is freely generated by the Atiyah-Bott classes

$$H^{\star}\left(\mathrm{Bun}_{G,\bar{k}}^{\vartheta}, \overline{\mathbb{Q}}_{\ell}\right) = \overline{\mathbb{Q}}_{\ell}\left[a_1, \ldots, a_r\right] \otimes \bigwedge^{\star}\left[b_i^j\right]_{i=1,\ldots r, j=1,\ldots,2g} \otimes \overline{\mathbb{Q}}_{\ell}\left[f_1, \ldots, f_r\right]$$

Reference

1. Heinloth's Lecture Notes.
2. Heinloth-Schmitt's paper "The Cohomology Rings of Moduli Stacks of Principal Bundles over Curves" states and proves the result for general reductive G.
3. Atiyah-Bott's original papers deals with the analytic setting and G semisimple. They used equivariant cohomology to frame their results.