The Game of Chomp

Written by: Eva Elduque

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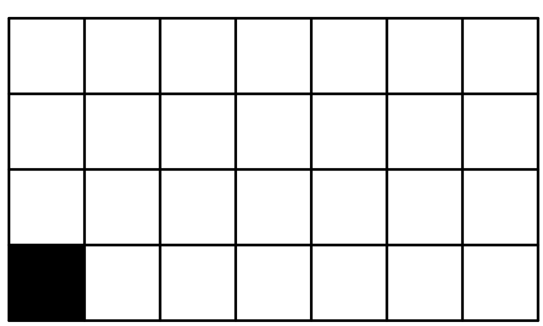
`https://www.math.wisc.edu/wiki/index.php/Madison_Math_Circle`.

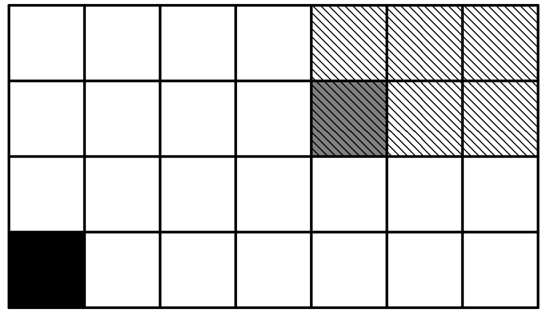
The Game:

Chomp is a two-player game played on a rectangular grid, or in a tastier version of the game, on a chocolate bar in which the bottom left square is poisoned. The chocolate bar can have as many rows and columns as we want. The two players take turns picking squares, and once they do, they remove (or eat) every square of the chocolate bar that is on top of the one they picked or to its right. The player who eats the poisoned square loses.

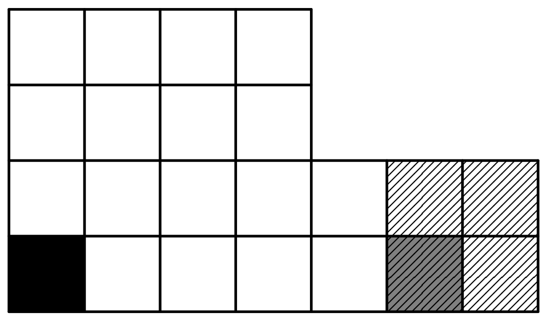
Let's do an example on a 4×7 grid. Here we have our chocolate bar where the bottom left square (colored in black) is poisoned.



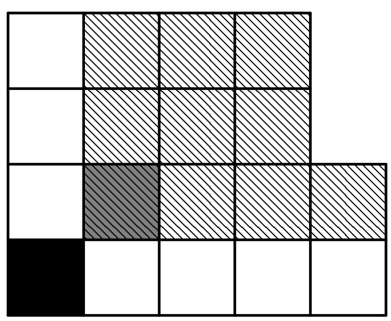
Alice and Bob are going to play a game of Chomp, and Alice is going to be the first one to move. She chooses the square that we have colored in gray, and eats all the striped squares.



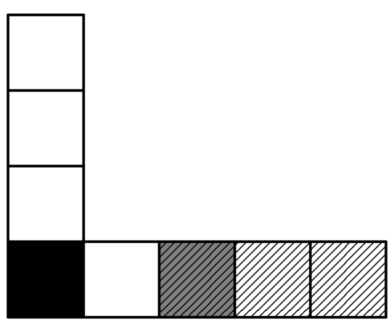
Now it's Bob's turn, and he picks the square $(6,1)$, that is, the square in the 6^{th} column and 1^{st} row (starting from the bottom left). He then eats everything to the top and right of it, which are the squares that we have filled with stripes going the other direction.



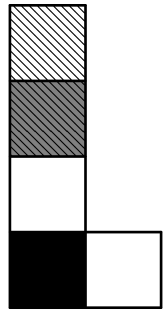
It's Alice's turn again, and she picks the square $(2,2)$.



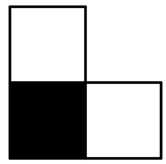
Bob now chooses the square $(3,1)$.



It is Alice's turn. She has five squares to pick from, and she picks the square $(1,3)$.



It's Bob's turn now, and the chocolate bar now looks like this.



If he picks the black square, he gets poisoned and loses. If he picks any of the white squares remaining, Alice, who really doesn't want to get poisoned, will pick the remaining white square, and Bob will be left with the poisoned square and lose. Hence, there's nothing Bob can do at this point to stop Alice from winning!

Exercise 1. Take grid paper, draw a rectangular board on it of however many rows and columns you like, find a partner, and play a few games of Chomp. **Tip:** to get better at playing Chomp, it helps to start with a small number of rows and columns. That way you will get a better understanding of what's going on.

Exercise 2. It was very smart of Alice to pick the square $(1,3)$ when she did it in her turn. In fact, if she had chosen any of the other 4 possible squares in that turn, Bob could have won regardless of what Alice did after that. Think about all these possible moves and about what Bob could have done to win for each of them.

Solution. If Alice had picked the black square, she would have automatically lost, and Bob would have won.

If Alice had picked the square $(1,4)$, then Bob could have picked $(1,3)$ and the situation this time would be the same as after Alice picked $(1,3)$ in our example (where Alice won), except that now the order in which Bob and Alice play is reversed. We've already analyzed this situation, and thus we know what Bob can do to win.

If Alice had picked the square $(1,2)$, then Bob could have picked $(2,1)$ and she would have been left with the poisoned square, making Bob the winner.

Similarly, if Alice had picked the square $(2,1)$, all that Bob would have needed to do to win is then pick $(1,2)$, which would have left Alice with the poisoned square again. □

Winning Strategy:

In a two-player game like Chomp with players A and B, we say that player A has a **winning strategy** if, no matter what player B does, there is always a sequence of moves that player A can do to counter player B's moves and assure that player A wins. Those moves that player A can do to win generally depend on the ones that player B does.

Having a winning strategy is different from knowing what the winning strategy is and actually winning. For example, after Alice picked the square $(1,3)$ she had a winning strategy, because, as we explained, she could counter Bob's moves to assure she won. However, she could have lost if Bob picked a white square in his next turn, and if she picked the black one after Bob's turn.

If we made it this far, we probably want to learn how to be invincible at Chomp. To become Chomp masters, we need to know which player (first or second) has a winning strategy (if any), so that we can pick who starts. Hopefully, our misinformed friends won't know as much as us, they won't know which player has a winning strategy and won't mind if we pick our place.

The following two exercises are a bit hard, so if you get stuck while trying to do them, it doesn't matter, you can just move on. That doesn't mean you shouldn't try to do them. Sometimes, even if we don't arrive to the solution of a problem, just thinking about it gives us a lot of insight about it!

Exercise 3. Does the second player have a winning strategy, provided that the chocolate bar has more than one square? **Hint:** If the second player had a winning strategy for Chomp, they would be able to counter **any** of the possible moves that the first player made. In particular, they would have a winning strategy if the first player removed the square at the top right corner. Think about this situation.

Solution. We are going to show that the second player doesn't have a winning strategy for any game of Chomp played on a rectangular grid of any dimension other than 1×1, and we are going to do this by contradiction. What this means is that we are going to assume that the second player has a winning strategy, and arguing from there we will reach an impossible situation, which will tell us that our assumption (the second player has a winning strategy) was wrong.

Suppose that the second player has a winning strategy for the game of Chomp with n rows and m columns, where n and m are not both 1. If the first player removes the square at the top right corner, then the second player can play their winning strategy and pick some other square, which we will call S, leaving the remaining chocolate bar in a shape that we call L. An example of this is shown in the following drawings, where $n = 4$ and $m = 7$.

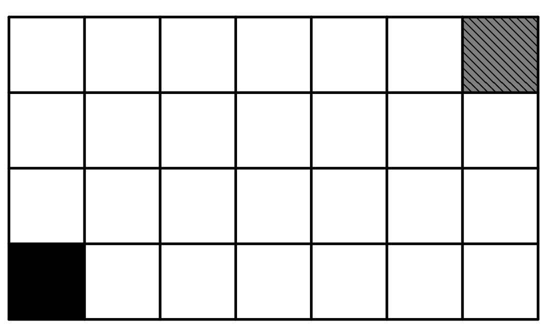


Figure 1. The first player picks and removes the square at the top right corner.

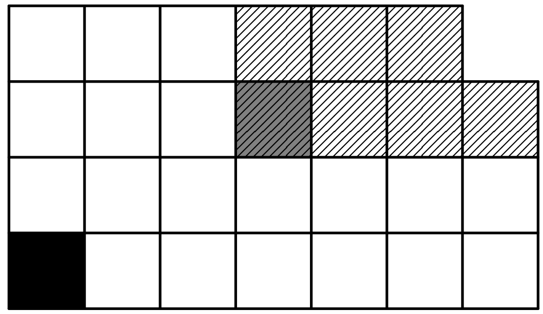


Figure 2. The second player picks the square S, colored in gray, and removes the striped area.

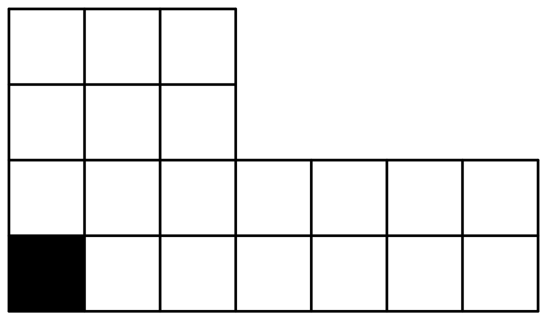


Figure 3. It's the first player's turn now, and this is what the chocolate bar looks prior to their turn, in shape L.

Since the second player has a winning strategy in the $n \times m$ game of Chomp, and they got to shape L after playing said strategy, we conclude that **the second player of a game of Chomp played on a chocolate bar of shape L has a winning strategy**. Now, imagine that the first player had picked the square S in their first turn. Then, the second player would have been left with the chocolate bar in shape L prior to their first turn. This is the same situation as we had before, but now the order of the players after they've reached shape L is reversed, so, since the second player of a game of Chomp on an $n \times m$ chocolate bar has a winning strategy, we conclude that **the first player of a game of Chomp played on a bar of shape L also has a winning strategy**. We've now reached an impossible situation, since the last two statements in bold letters can't be true at the same time. Hence, our assumption (the second player has a winning strategy) is false. □

If you have done and understood the previous exercise correctly, you now know that **the second player in a rectangular game of Chomp doesn't have a winning strategy**.

Exercise 4. Does the first player have a winning strategy? **Hint:** The second player doesn't have a winning strategy, so there is a move that the first player can play such that there's no possible move that the second player can do to ensure a second player win.

Solution. Suppose that the first player plays that first move that we mentioned in the hint. After the first move by both players, we are left with a (possibly non-rectangular) game of Chomp in which the second player doesn't have a winning strategy, and the board now has less squares. Now, we can repeat this argument to find a move that the first player can play in their second turn such that there's no possible move that the second player can do to ensure a second player win. Note that this move in the first player's second turn will depend on what the second player did in their first turn. After both player's second turn, we are left with a board with even less squares in which the first player has a move such that there's no possible move that the second player can do to ensure a second player win. If we keep repeating this argument, we will eventually run out of squares on the board, and since at no time can the second player assure a win, the first player will end up winning, no matter what the second player does. This means that we have proved that the first player has a winning strategy. □

If you have done and understood the previous exercise correctly, you now know that **the first player in a rectangular game of Chomp has a winning strategy**. Now that we know that, we definitely want to be the first player in a game against our friends! Unfortunately, as we have mentioned before, knowing that the first player has a winning strategy **doesn't mean we will win if we play first, because we don't know what the winning strategy actually is**, even if we know it exists!

The sad truth is that it is very hard to know what the specific winning strategy for an $n \times m$ game of Chomp is in general. If the values of n and m aren't very big, then a computer can easily calculate all the possible outcomes of every move and always beat a human if the computer goes first, but humans aren't able to do all those computations as fast. You can check how hard it is to beat a computer at Chomp in `https://www.math.ucla.edu/~tom/Games/chomp.html`, even if you are the first player!

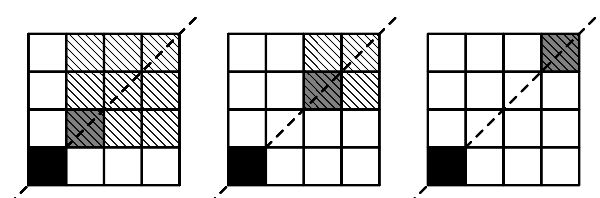
On the bright side, there are specific values of n and m for which we can deduce the winning strategy and make sure we beat our friends if we go first! This is what we are going to do next. Hopefully, they won't mind if we pick the dimensions of the chocolate bar!

How to win in certain cases

From now on, since we know that the first player has a winning strategy, we will refer to the first player in first person, because we want to be the ones who win!

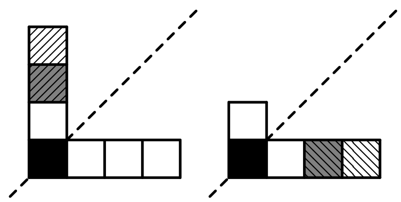
Exercise 5. Deduce a winning strategy for the first player in the case where the chocolate bar is an $n \times n$ square (with the same number of rows and columns), where $n > 1$. **Hint:** After your first move, you will want to "mirror" in some way what the second player does.

Solution. For our first move, we want to pick a square that leaves the chocolate bar symmetric with respect to the diagonal of the square that goes from the bottom left corner to the top right one. In fact, after each of our moves, we are going to want to leave the chocolate bar symmetric in this way. We will use this symmetry in the following moves. The only possible first moves that allow us to leave the board with this kind of symmetry are picking any of the squares in that diagonal (except for the poisoned one, of course!). We draw all of these possible moves in the case where $n = 4$ to give a better idea of what we said. Note that all of the remaining boards after any of these moves are symmetric with respect to the diagonal marked with a dashed line.

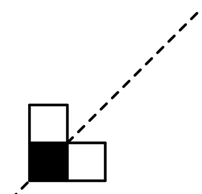


However, **only one of these moves is a good first move**. We said that we wanted to leave the chocolate bar symmetric with respect to the dashed diagonal after each one of our moves, but if we pick any other square in the diagonal other than $(2,2)$ for our first move, the second player could pick the square $(2,2)$ in their first turn and there would be nothing we could do to leave the chocolate bar symmetric after our second turn. Hence, **we want to pick the square $(2,2)$ in our first turn**. This situation corresponds to the first of the drawings above.

Now, we just want to mirror what the second player does. That is, if the second player now picks the square $(1,l)$, we will pick the square $(l,1)$ in our next turn. That way, after every single one of our turns, the board will be left symmetric with respect to the dashed diagonal. For example, if $n = 4$ again, and we have picked the square $(2,2)$ in our first move (as we should), and the second player picks the square $(1,3)$, we will pick the square $(3,1)$ in our next turn. The second player's move and the way we mirror it in this example are shown in the following picture.



These moves leave the board looking like this, symmetric with respect to the dashed diagonal, and not letting the second player make a move that keeps the symmetry.



In this way, we can keep mirroring what the second player does, and we will always be able to do so because after all of our turns, the board is left being symmetric with respect to the dashed diagonal. But after each turn, there will be less squares on the board than before, so eventually the second player will have to pick the poisoned square and we will have won! ☐

Exercise 6. Deduce a winning strategy for the first player in the case where the chocolate bar is a $2 \times n$ board (2 rows and n columns), where $n \geq 1$.

Solution. Let's start by picking the square at the top right corner for our first move. The strategy we are going to follow is, that after each and every one of our turns, the board looks like a $2 \times l$ board with the square at the top right corner removed. Let's see that we can do that.

If, after our first turn, our opponent picks a square from the bottom row, that move will leave the board looking like a rectangular grid with two rows, so we just have to pick the square at the top right corner of the remaining board.

If, after our first turn, our opponent picks a square from the top row, which will have coordinates of the form $(l,2)$, with $l < n$, then we will pick the square in the bottom row just to the left of the square they picked, that is, the square $(l+1,1)$. This will leave the board looking like a $2 \times l$ rectangular grid with the square at the top right removed.

Now, we can keep repeating this process. Note that every time we play, the board keeps getting shorter, so at some point after one of our turns (if the second player hasn't picked the poisoned square in the previous rounds) we will be left with a 2×1 board with the square at the top right removed, or equivalently, the second player will be left with the poisoned square and we will win! □

The following exercise shows that it can only take one little mistake to have the other player steal your winning strategy, so you have to be very careful while playing!

Exercise 7. In a game of Chomp on a $2 \times n$ board, the second player has a winning strategy if the first player's first pick is anything other than the square at the top right corner. Can you explain why?

Solution. If we picked any square in the bottom row, what would be left of the board would be a rectangular grid, and it would be like our opponent, who started out being the second player, was the first player in that new game of rectangular Chomp. Thus, our opponent would have a winning strategy from then on by what we did in Exercise 4. Hence, to assure we win, our first move has to be something in the top row.

Now, assume that our first move as first player is some square in the top row other than the top right corner, that is, we pick the square $(l, 2)$, with $l < n$. After that, the second player could pick the square in the bottom row just to the left of the square we picked, that is, the square $(l+1, 1)$. Note that the second player can do this because $l + 1 \geq n$. After the second player's turn, the board would be looking like a $2 \times l$ game of Chomp in which our opponent was the first player and had picked the square on the top right corner for their first turn. By Exercise 6, we know that our opponent has a winning strategy in this case too.

Hence, the only move we can do as first players that doesn't automatically give a winning strategy to our opponent is picking the square at the top right corner. □

References

[1] http://www.math.cornell.edu/~mec/2003-2004/graphtheory/chomp/howtoplaychomp.html