Continuum Hypothesis: On Platonism and Pluralism

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Outline

Introduction

Axiom

(Continuum Hypothesis) $2^{\aleph_0} = \aleph_1$.

Theorem

(Gödel) $\mathrm{ZFC} + V = L \vdash \mathrm{CH}$.

Theorem

(Cohen) *There is a forcing extension* $V[G]$ *such that* $V[G] \vDash \mathrm{ZFC} + \neg\mathrm{CH}$.

Introduction

Corollary

CH is independent of ZFC.

Question. Why is CH still a meaningful question?

A. Mathematically useful and philosophically meaningful.

Approaching the Problem

The naive attempt to settle CH goes by finding "the complete theory of V" and showing CH (or $\neg$CH) as a consequence of it.

Problem 1: V is too large.

Solution 1: Focus on "lower levels" of V: $V_{\omega+2}$, or $H(\omega_2)$ (where $H(\alpha)$ is the set of all sets which hereditarily have cardinality $<\alpha$).

Problem 2: Incompleteness theorem.

Solution 2: Weaken the notion of completeness.

Negative Evidence

Candidates for completeness:

- Empirically Complete
- (Effectively) Forcing Complete (under large cardinals)
- Complete in Ω-logic: roughly expressing forcing-invariant truths

Ω-logic is sound and forcing-invariant (under large cardinals).

Goal: Search for a sentence φ such that: for every sentence θ,

$$\mathrm{ZFC} + \varphi \vdash_\Omega \text{``}(H(\omega_2), \in) \models \theta,\text{'' or } \mathrm{ZFC} + \varphi \vdash_\Omega \text{``}(H(\omega_2), \in) \models \neg\theta.\text{''}$$

Negative Evidence

It turns out that such axioms exist: the (*) axiom is one that is maximal in a certain sense.

Theorem

$\mathrm{ZFC} + (*) \vdash 2^{\aleph_0} = \aleph_2$.

In fact, even though all such axioms may lead to different theories of $(H(\omega_2), \in)$, they settle CH in the same way as above.

Large Cardinals and Inner Model Program

Recall that $V = L$ arose in the consistency proof of ZFC. It decides a lot more (common) statements in set theory, such as GCH.

Question: Should we accept $V = L$?

Answer: No. Not enough to accommodate large cardinals.

Question: Should we get rid of L?

Answer: No. We prefer to work with "L-like" inner models.

However, it is difficult to get a canonical global theory: these inner models are specialized for the cardinal, so one needs to keep going up…

Positive Evidence

…until the level of supercompact cardinals.

No inner models have reached such a level yet, but we have an idea of what it should look like. This leads to the axiom $V = \text{Ultimate-}L$.

Theorem

$\text{ZFC} + V = \text{Ultimate-}L$ implies:

(1) CH; (2) $V = \text{HOD}$; (3) Ω-logic is complete.

And $V = \text{Ultimate-}L$ could be potentially generalized to decide more independent statements.

Indeterminate Evidence

First, we already know too much consequences of both CH and ¬CH, so that accepting either would result in loss of knowledge on the other side.

Second, we can consider all models of ZFC as possible worlds, with forceability being the accessibility relation. (This leads to the modal system $S4.2$.) Here, both CH and ¬CH are necessarily possible.

Adding to the "many-worlds interpretation," one can actually formalize what it is like to be in a "set-theoretic multiverse," resulting in the Multiverse Axioms.

"Toy Models" of the Multiverse Axioms: All countable, computably saturated models of ZFC.

Conclusion

This has almost become a philosophical debate, where mathematical arguments serve as prerequisites. However, mathematical evidence can still influence one's beliefs.

Remark: Platonism and pluralism are not completely opposite here. In fact, in defining Ω-logic, different forcing extensions had to be considered; and conversely, evidence suggests that multiverses have "cores" which model $V = \text{Ultimate-}L$.

So the matter might just be a difference in scope.

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Thank you for listening!