Comments Math 722 & 725 versions of the
QUALIFYING EXAM in ANALYSIS

UW-Madison, August 24, 2016

Common questions

Problem 1. For $n \geq 2$ an integer, define $F(n)$ to be the function $F(n) = \max\{k \in \mathbb{Z} : 2^k/k \leq n\}$. Does $\sum_{n=2}^{\infty} 2^{-F(n)}$ converge or diverge?

Discussion – first solution. For any $n \geq 2$ we have

$$\frac{2^{F(n)}}{F(n)} \leq n < \frac{2^{F(n)+1}}{F(n)+1}.$$

Taking base-2 logarithms ($\lg x = \log_2 x$) we find

$$F(n) - \lg F(n) \leq \lg n < F(n) + 1 - \lg\bigl(F(n)+1\bigr).$$

Since $\lg x = o(x)$ for $x \to \infty$, we have $\lg F(n) = o(F(n))$, so that $F(n) = \lg n + o(F(n))$, which implies $F(n) = \lg n + o(\lg n)$ as $n \to \infty$.

Thus we can estimate the terms in our sum by

$$2^{-F(n)} = \frac{1}{nF(n)} = \frac{1}{n(1+o(1))\lg n} = \frac{1+o(1)}{n \lg n} \qquad (n \to \infty).$$

Since the series $\sum (n \lg n)^{-1}$ diverges, the series $\sum 2^{-F(n)}$ also diverges.

Second solution. By definition $k = F(n)$ holds if and only if

$$\frac{2^k}{k} \leq n < \frac{2^{k+1}}{k+1}.$$

For any given integer $k \in \mathbb{N}$ there are therefore exactly

$$\left\lfloor \frac{2^{k+1}}{k+1} \right\rfloor - \left\lfloor \frac{2^k}{k} \right\rfloor$$

values of n such that $F(n) = k$. Therefore

$$\begin{aligned}
\sum_{n=2}^{\infty} 2^{-F(n)} &= \sum_{k=F(2)}^{\infty} 2^{-k} \left(\left\lfloor \frac{2^{k+1}}{k+1} \right\rfloor - \left\lfloor \frac{2^k}{k} \right\rfloor \right) \\
&\geq \sum_{k=F(2)}^{\infty} 2^{-k} \left(\frac{2^{k+1}}{k+1} - \frac{2^k}{k} - 1 \right) \\
&= \sum_{F(2)}^{\infty} \left(\frac{2}{k+1} - \frac{1}{k} - 2^{-k} \right) \\
&= \sum_{F(2)}^{\infty} \left(\frac{k-1}{k(k+1)} - 2^{-k} \right)
\end{aligned}$$

The geometric series $\sum 2^{-k}$ converges, but the series $\sum \frac{k-1}{k(k+1)}$ diverges because $\frac{k-1}{k(k+1)} \geq \frac{C}{k}$ for some constant C.

Hence the series $\sum_{n \geq 2} 2^{-F(n)}$ also diverges.

Problem 2. Let a_n be a sequence of complex numbers. Suppose that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} a_i = L,$$

and

$$\sum_{n=1}^{\infty} n|a_{n+1}-a_n|^2 < \infty.$$

Prove that $a_n \to L$.

Discussion. For any sequence a_n it can happen that the averages $b_n = (a_1+\cdots+a_n)/n$ converge even though the a_n themselves do not converge (e.g. think of $a_n = (-1)^n$: averaging reduces the oscillations the sequence a_n may have). The second piece of information in the problem provides a bound for the size of the oscillations in the sequence a_n. In the following we show that the average of $\{a_{n+1},\ldots,a_{2n}\}$ also converges to L by writing it in terms of the averages of $\{a_1,\ldots,a_n\}$ and of $\{a_1,\ldots,a_{2n}\}$. We then use the fact that $\sum k|a_{k+1}-a_k|^2 < \infty$ to show that the average of $\{a_{n+1},\ldots,a_{2n}\}$ is approximately equal to a_n itself, so that $a_n \to L$ follows.

We have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=n+1}^{2n} a_k = \lim_{n\to\infty}\left\{2\cdot\frac{1}{2n}\sum_{k=1}^{2n}a_k - \frac{1}{n}\sum_{k=1}^{n}a_k\right\} = 2L - L = L.$$

For $n+1 < k \le 2n$ we also have

$$a_k - a_n = \sum_{i=n+1}^{k} a_i - a_{i-1}$$

so that

$$\left|\frac{\sum_{n+1}^{2n} a_k}{n} - a_n\right| = \left|\frac{\sum_{n+1}^{2n}(a_k - a_n)}{n}\right| = \frac{1}{n}\left|\sum_{n+1}^{2n}\sum_{n+1}^{k}(a_i - a_{i-1})\right|.$$

For each k we have

$$\left|\sum_{n+1}^{k}(a_i - a_{i-1})\right| \le \sum_{n+1}^{2n}|a_i - a_{i-1}|,$$

so

$$\left|\frac{\sum_{n+1}^{2n} a_k}{n} - a_n\right| \le \sum_{n+1}^{2n}|a_i - a_{i-1}|.$$

Applying Cauchy's inequality we find

$$\left|\frac{\sum_{n+1}^{2n} a_k}{n} - a_n\right| \le \sqrt{\sum_{n+1}^{2n}\frac{1}{i}}\sqrt{\sum_{n+1}^{2n} i|a_i - a_{i-1}|^2} \le \sqrt{\sum_{n+1}^{\infty} i|a_i - a_{i-1}|^2}$$

where we have used that $\sum_{n+1}^{2n}\frac{1}{i} < \sum_{n+1}^{2n}\frac{1}{n} = 1$. Since we are given that the series $\sum i|a_i - a_{i-1}|^2$ converges, we conclude that

$$\lim_{n\to\infty}\left|\frac{\sum_{n+1}^{2n} a_k}{n} - a_n\right| = 0.$$

Hence $\lim a_n = L$.

Problem 3. Let $0 < \alpha < 1$. A function $f \in C([0,1])$ is said to be Hölder continuous of order α if there is a constant $C > 0$ such that

$$|f(x) - f(y)| \le C|x - y|^\alpha$$

for all $x, y \in [0,1]$. Show that for every $0 < \alpha < 1$, the function

$$f(x) = \sum_{n=1}^{\infty} 2^{-n\alpha} \cos(2^n x)$$

is Hölder continuous of order α.

Discussion. In a direct attack on the problem one would try to estimate

$$f(x) - f(y) = \sum_{n=1}^{\infty} 2^{-n\alpha}\big(\cos(2^n x) - \cos(2^n y)\big)$$

by using

$$|\cos A - \cos B| \le |A - B|, \text{ and thus } |\cos 2^n x - \cos 2^n y| \le 2^n |x - y|,$$

which follows from the mean value theorem and the fact that $|\sin\theta| \le 1$. This leads to

$$|f(x) - f(y)| \le \sum_{n=1}^{\infty} 2^{-n\alpha}\big|2^n x - 2^n y\big| = \sum_{1}^{\infty} 2^{(1-\alpha)n}|x - y|.$$

Unfortunately the series diverges because $1 - \alpha \ge 0$, so this does not lead to anything. Looking back we see that the estimate

$$|\cos 2^n x - \cos 2^n y| \le 2^n |x - y|$$

is wasteful when n is large, and can be improved by observing

$$|\cos 2^n x - \cos 2^n y| \le |\cos 2^n x| + |\cos 2^n y| \le 2.$$

We now have two estimates for $|\cos 2^n x - \cos 2^n y|$, one of which is better for small n while the other is better for large n. Let $N \in \mathbb{N}$ be some integer, to be chosen below. Then we can estimate $f(x) - f(y)$ by

$$\begin{aligned}
|f(x) - f(y)| &\le \sum_{1}^{N} 2^{-n\alpha} 2^n |x - y| + \sum_{N+1}^{\infty} 2^{-n\alpha} \cdot 2 \\
&\le \sum_{-\infty}^{N} 2^{(1-\alpha)n}|x - y| + 2 \cdot \sum_{N+1}^{\infty} 2^{-n\alpha} \qquad \text{(sum the two geometric series)} \\
&= 2 \cdot \frac{2^{N(1-\alpha)}}{1 - 2^{\alpha - 1}}|x - y| + 2\frac{2^{-(N+1)\alpha}}{1 - 2^{-\alpha}} \\
&\le C_\alpha \big\{2^N |x - y| + 1\big\} 2^{-N\alpha}.
\end{aligned}$$

Here C_α is a constant that only depends on α. We now choose N so that

$$2^N |x - y| \le 1 < 2^{N+1}|x - y|,$$

and conclude that indeed

$$|f(x) - f(y)| \le C'_\alpha |x - y|^\alpha.$$

Problem 4. Let $f : [0,1] \to \mathbb{R}$ be continuous with

$$\min_{0 \le x \le 1} f(x) = 0.$$

Assume that for all $0 \le a \le b \le 1$ we have

$$\int_a^b [f(x) - \min_{a \le y \le b} f(y)]dx \le \frac{|b-a|}{2}.$$

Prove that for all $\lambda \ge 0$ we have

$$|\{x : f(x) > \lambda + 1\}| \le \tfrac{1}{2}|\{x : f(x) > \lambda\}|.$$

Here, $|S|$ denotes the Lebesgue measure of a set S.

Discussion. Let $\lambda > 0$ be given, and consider $E_\lambda = f^{-1}\big((\lambda, \infty)\big)$. Clearly $E_{\lambda+1} \subset E_\lambda$. The problem asks us to show that $|E_{\lambda+1}| \le \frac{1}{2}|E_\lambda|$.

The set $E_\lambda \subset [0,1]$ is open, and therefore is the union of a countable number of intervals:

$$E_\lambda = \cup_i I_{\lambda i}.$$

On each interval $I_{\lambda i}$ we have $\inf_{I_{\lambda i}} f(x) = \lambda$, because $\min_{0 \le x \le 1} f(x) = 0$, and because f is continuous. Thus on each $I_{\lambda i}$ we have

$$|E_{\lambda+1} \cap I_{\lambda i}| \le \int_{E_{\lambda+1} \cap I_{\lambda i}} \big(f(x) - \lambda\big)dx \le \int_{I_{\lambda i}} \big(f(x) - \lambda\big)dx \le \frac{1}{2}|I_{\lambda i}|.$$

Thus

$$|E_{\lambda+1}| = \sum_i |E_{\lambda+1} \cap I_{\lambda i}| \le \sum_i \frac{1}{2}|I_{\lambda i}| = \frac{1}{2}|E_\lambda|,$$

which is what we had to show.

Problem 5. Give an example of a non-empty closed subset of $L^2([0,1])$ that does not contain a vector of smallest norm. Prove your assertion.

Discussion. Let ϕ_n be any orthonormal set of functions in $L^2([0,1])$, e.g. $\phi_n(x) = \sqrt{2}\sin n\pi x$. Then the set

$$C = \left\{\tfrac{n+1}{n}\phi_n : n \in \mathbb{N}\right\}$$

is closed, nonempty, and does not contain a function with least norm.

To see that C is closed, note that by Pythagoras

$$\|\phi_n - \phi_m\|_2 = \sqrt{\Big(\frac{n+1}{n}\Big)^2 + \Big(\frac{m+1}{m}\Big)^2} \ge \sqrt{2},$$

so all points in C are isolated, and C must indeed be closed.

On the other hand

$$\inf_{n \in \mathbb{N}} \|\phi_n\|_2 = \inf_{n \in \mathbb{N}} \frac{n+1}{n} = 1$$

while $\|\phi_n\| > 1$ for all $n \in \mathbb{N}$.

Problem 6. Let f be a non-negative measurable function on $[0,1]$. Prove that the following statements are equivalent.

(1) There exists $a > 0$ such that

$$\int_0^1 e^{af(x)} dx < \infty.$$

(2) There exists $C > 0$ such that

$$\left(\int_0^1 f(x)^p dx\right)^{1/p} \le Cp \quad \text{for all} \quad 1 \le p < \infty.$$

Discussion: (1) $\implies$ (2)**.** For any integer $n \in \mathbb{N}$ and any $x \ge 0$ we have

$$\frac{x^n}{n!} \le 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = e^x.$$

Hence

$$\int_0^1 f(x)^n dx \le n! a^{-n} \int_0^1 e^{af(x)} dx.$$

Take the n^{th} root:

$$\|f\|_n \le C(n!)^{1/n} < Cn,$$

because $n! = 1 \cdot 2 \cdots n < n^n$.

Thus we have shown $\|f\|_n \le Cn$ for all positive integers n. For any non integer $p \in [1, \infty)$ we choose $n \in \mathbb{N}$ so that $p \le n$ and use $\|f\|_p \le \|f\|_n$. The latter inequality follows from Hölder's inequality:

$$\|f\|_p^p = \int_0^1 f(x)^p \, dx \le \left(\int_0^1 f(x)^n \, dx\right)^{\frac{p}{n}} \left(\int_0^1 1^{\frac{n}{n-p}} \, dx\right)^{1-\frac{p}{n}} = \|f\|_n^p.$$

In this derivation we used $n! < n^n$, which is the simplest approach Alternatively, one can use Stirling's approximation,

$$n! = \sqrt{2\pi n} \, n^n e^{-n+o(1)} \qquad (n \to \infty),$$

which implies

$$(n!)^{1/n} = (1 + o(1))^{1/n} \, (2\pi n)^{1/n} \frac{n}{e} \le Cn.$$

A variation on this solution. Begin with the observation that $1 + x \le e^x$ for all $x \ge 0$, and thus

$$\left(\frac{x}{p}\right)^p \le \left(1 + \frac{x}{p}\right)^p \le \left(e^{x/p}\right)^p = e^x.$$

This implies

$$\frac{1}{p^p} \|f\|_p^p = a^{-p} \int \left|\frac{af(x)}{p}\right|^p dx \le a^{-p} \int e^{af(x)} dx,$$

and thus

$$\|f\|_p \le \frac{p}{a} \int e^{af(x)} dx.$$

In this approach p is not restricted to integer values.

(2) $\Longrightarrow$ (1). To prove the converse we expand $e^{af(x)}$ in a power series and compute

$$\int_0^1 e^{af(x)}dx = \sum_{k=0}^{\infty} \frac{a^n}{n!}\int_0^1 f(x)^n dx \le \sum_{k=0}^{\infty} \frac{a^n}{n!} C^n n^n.$$

The series converges if $a > 0$ is small enough, e.g. by looking at the ratio between consecutive terms

$$\lim_{n\to\infty} \frac{\frac{a^{n+1}}{(n+1)!}C^{n+1}(n+1)^{n+1}}{\frac{a^n}{n!}C^n n^n} = \lim_{n\to\infty} aC\Big(\frac{n+1}{n}\Big)^n = eaC.$$

Thus the series converges when $a < (eC)^{-1}$; for those a we then also have

$$\int_0^1 e^{af(x)}dx < \infty.$$

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Problem 7. Suppose that $|a| \ne 1$. Compute the integral

$$\int_0^{2\pi} \frac{\cos^2 3\phi}{1 - 2a\cos\phi + a^2} d\phi$$

Discussion. We substitute $t = e^{i\phi}$, which leads to

$$\begin{aligned} I_a = \int_0^{2\pi} \frac{\cos^2 3\phi}{1 - 2a\cos\phi + a^2} d\phi &= \int_{\mathcal{C}} \frac{(\frac12)^2\big(t^3 + t^{-3}\big)^2}{1 - a\big(t + t^{-1}\big) + a^2}\frac{dt}{it} \\ &= \frac{1}{4i}\int_{\mathcal{C}} \frac{t^6 + 2 + t^{-6}}{t - at^2 - a + a^2 t} dt \\ &= \frac{1}{4i}\int_{\mathcal{C}} \frac{t^6 + 2 + t^{-6}}{(t-a)(1-at)} dt \end{aligned}$$

where C is the unit circle, traversed counterclockwise. We can find the integral by computing the residues of the poles of the integrand that lie inside the unit circle. There are two poles, namely, $t = 0$ and $t = a$ or $t = 1/a$, depending on whether $|a| < 1$ or $|a| > 1$ (the case $a = 0$ should be treated separately).

We can avoid computing the residue at $t = 0$ by observing that

$$\frac{1}{4i}\int_{\mathcal{C}} \frac{t^{-6}}{(t-a)(1-at)} dt = \frac{1}{4i}\int_{\mathcal{C}} \frac{r^6}{(r-a)(1-ar)} dr,$$

which follows by substituting $t = 1/r$, and which is perhaps easier to see from the original unexpanded form of the integral

$$\int_{\mathcal{C}} \frac{t^6}{1 - a\big(t + t^{-1}\big) + a^2}\frac{dt}{it} = \int_{\mathcal{C}} \frac{r^{-6}}{1 - a\big(r + r^{-1}\big) + a^2}\frac{dr}{ir}.$$

Note that $dt/t = -dr/r$, and that the transformation $t \mapsto 1/t$ reverses the orientation of the path C.

The upshot is

$$I_a = \frac{1}{4i}\int_{\mathcal{C}} \frac{2t^6 + 2}{(t-a)(1-at)} dt = \frac{1}{2i}\int_{\mathcal{C}} \frac{t^6 + 1}{(t-a)(1-at)} dt.$$

If $|a| < 1$ then

$$I_a = 2\pi i \operatorname{Res}\Big(\frac{1}{2i}\frac{t^6+1}{(t-a)(1-at)}, t=a\Big) = \pi\frac{1+a^6}{1-a^2}.$$

If $|a| > 1$ then

$$I_a = 2\pi i \operatorname{Res}\Big(\frac{1}{2i}\frac{t^6+1}{(t-a)(1-at)}, t=1/a\Big) = \pi\frac{1+a^{-6}}{1-a^2}.$$

Problem 8. Suppose that $f \in C^2([0,1])$ is real valued and satisfies $|f(1)| > |f(0)|$. Prove that

$$F(z) = \int_0^1 f(t)\sin(tz)dt$$

has infinitely many real roots and only finitely many strictly complex roots.

Discussion. Integrate by parts, twice:

$$F(z) = \Big[-f(t)\cos(tz)/z + f'(t)\sin(tz)/z^2\Big]_{t=0}^1 - \int_0^1 f''(t)\frac{\cos tz}{z^2}dt.$$

Hence

$$zF(z) = f(0) - f(1)\cos z + f'(1)\frac{\sin z}{z} - \frac{1}{z}\int_0^1 f''(t)\cos tz\,dt.$$

This identity shows that for large z the function $zF(z)$ is a close to

$$G(z) = f(0) - f(1)\cos z,$$

and thus its zeros will also be close. We can use Rouché's theorem (the winding number on the boundary determines the number of zeros in a region) to make this precise.

To estimate the size of $\sin z$ and $\cos z$ when z is complex we use the addition formulas

$$\sin(x+iy) = \sin x\cosh y + i\sinh x\cos y, \qquad \cos(x+iy) = \cos x\cosh y - i\sin x\sinh y$$

which, in view of

$$\sin^2 x + \cos^2 x = \cosh^2 y - \sinh^2 y = 1,$$

imply

$$|\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y$$
$$|\cos(x+iy)|^2 = \cos^2 x + \sinh^2 y.$$

Step 1. With these identities we now first show that $zF(z)$ has no roots $z = x+iy$ with $|y| \geq A$, for some A to be determined.

If $|y| \geq A$, then

$$|G(z)| \geq |f(1)|\,|\cos(x+iy)| - |f(0)| \geq |f(1)|\,|\sinh y| - |f(0)|.$$

Furthermore, $|y| \geq A$ also implies

$$|zF(z) - G(z)| \leq \frac{1}{|z|}\Big\{|f'(1)|\cosh y + \sup_{0\leq t\leq 1}|f''(t)|\cosh y\Big\} \leq \frac{C}{A}\cosh y,$$

where C is a constant that only depends on the function f. We now choose A so large that

$$|f(1)|\,|\sinh y| - |f(0)| > \frac{C}{A}\cosh y$$

whenever $|y| \geq A$. If A is chosen this way, then we have

$$|zF(z)| \geq |G(z)| - |zF(z) - G(z)| > 0$$

for all $z = x + iy$ with $|y| \geq A$. Thus all roots of $zF(z)$ lie in the strip $-A < y < A$.

Step 2. Consider the sequence of rectangles

$$R_n = \{z = x + iy \in \mathbb{C} : n\pi < x < (n+1)\pi, |y| < A\}.$$

We have

$$|G(n\pi + iy)| = |f(0) - (-1)^n f(1)\cosh y| \geq |f(1)|\cosh y - |f(0)| \geq |f(1)| - |f(0)|,$$

so that $|f(1)| > |f(0)|$ implies that $G(z) \neq 0$ on the boundary of R_n. The number of zeros of $G(z)$ therefore does not change if one changes the parameters $f(0)$, $f(1)$, and we conclude that $G(z)$ has the same number of zeros as $\cos z$ in the rectangle R_n, namely, exactly one.

On the vertical side $z = n + iy$, $|y| \leq A$, of R_n we have

$$\begin{aligned} |zF(z) - G(z)| &\leq \frac{1}{|z|}\Big\{|f'(1)|\cosh y + \sup_{0\leq t\leq 1} |f''(t)|\cosh y\Big\} \\ &\leq \frac{C}{n}\cosh A. \end{aligned}$$

For

$$n > \frac{C\cosh A}{|f(1)| - |f(0)|}$$

we therefore have

$$|zF(z) - G(z)| < |f(1)| - |f(0)| \leq |G(z)|.$$

By Rouché's theorem the functions $zF(z)$ and $G(z)$ have the same number of zeros in R_n, i.e. $F(z)$ has exactly one zero in R_n. Since $f(t)$ is real valued, we have $F(\bar{z}) = \overline{F(z)}$, so the unique zero of $F(z)$ in R_n must be real.

Problem 9. Let $f(z)$ be the function defined for $\operatorname{Re} z > 0$ by $f(z) = \sum_{n=1}^{\infty} \exp(-n!z)$. Prove that f cannot be analytically continued to any connected open subset of the complex plane that strictly contains $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Discussion. Any open connected subset Ω of the complex plane that strictly contains the right half plane must contain an open subset of the imaginary axis, and thus it must contain a point of the form $2\pi ri$ with $r \in \mathbb{Q}$. If f can be extended to an analytic function on Ω then $f(z)$ will have a limit (namely $f(2\pi ri)$) as z approaches $2\pi ri$ from within the right half plane. We will show that this limit does not exist, so that f cannot be extended to Ω.

Let $r = \frac{p}{q}$ and consider the fact that for $z = x + 2\pi\frac{p}{q}i$ with $x > 0$ we can write $f(z)$ as

$$f(z) = \sum_{n=1}^{q} e^{-n!z} + \sum_{n=q+1}^{\infty} e^{-n!x}.$$

The first terms represent an entire function of z, while the remaining series is real valued if $x > 0$ is real. By monotone convergence we have

$$\sum_{n=q+1}^{\infty} e^{-n!x} \to \infty \text{ as } x \searrow 0.$$

This shows that although the holomorphic function $f(z)$ is defined on the right half plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, the function $f(z)$ has no limit as z approaches $2\frac{p}{q}\pi i$. The function therefore has no analytic continuation beyond the right half plane.

Problem 7.

A. For which $m \in \mathbb{Z}$ does there exist a tempered distribution $\Lambda_m \in \mathcal{S}'(\mathbb{R}^d)$ such that Λ_m agrees with $|x|^{-m}$ on $\mathbb{R}^d \setminus \{0\}$? For each m, either prove that no such distribution exists or give an example of one.

B. Does there exist $\Lambda \in \mathcal{S}'(\mathbb{R})$ such that Λ agrees with $e^{1/|x|}$ on $\mathbb{R} \setminus \{0\}$?

Discussion – Part A. A linear functional $\Lambda : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ defines a tempered distribution if it is a continuous linear functional, i.e. if for some $k \in \mathbb{N}$ there is a $C_k < \infty$ such that for all test functions $\phi \in \mathcal{S}(\mathbb{R}^d)$ one has

$$|\langle \Lambda, \phi \rangle| \le C_k \|\phi\|_k,$$

where, by definition,

$$\|\phi\|_k = \sup_{x \in \mathbb{R}^d, |\alpha| \le k} \bigl(1 + |x|^2\bigr)^{k/2} |D^\alpha \phi(x)|.$$

Two tempered distributions $\Lambda, \tilde{\Lambda} \in \mathcal{S}'(\mathbb{R}^d)$ agree on some open set $\Omega \subset \mathbb{R}^d$ if $\langle \Lambda, \phi \rangle = \langle \tilde{\Lambda}, \phi \rangle$ holds for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp} \phi \subset \Omega$.

The most natural choice for Λ_m would be

$$\langle \Lambda_m, \phi \rangle = \int_{\mathbb{R}^d} \frac{\phi(x)}{|x|^m} \, dx.$$

However, if m is too large then the integral does not converge at $x = 0$.

To fix this we modify Λ_m by subtracting sufficiently many terms in the Taylor expansion of the test function.

Let $\eta \in C_c^\infty(\mathbb{R}^d)$ satisfy $\eta(x) = 1$ on a neighborhood of the origin. For any $N \in \mathbb{N}$ we now consider the integral

$$\langle \Lambda_{m,N}, \phi \rangle \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \Bigl\{ \phi(x) - \eta(x) \sum_{|\alpha| \le N} \frac{x^\alpha}{\alpha!} D^\alpha \phi(0) \Bigr\} \frac{dx}{|x|^m}$$

The integrand satisfies

$$\left| \Bigl\{ \phi(x) - \eta(x) \sum_{|\alpha| \le N} \frac{x^\alpha}{\alpha!} D^\alpha \phi(0) \Bigr\} |x|^{-m} \right| \le C \|\phi\|_{N+1} |x|^{N+1-m}.$$

Thus if N is large enough ($N \ge m - d$) then the integral converges, so that $\Lambda_{m,N}$ defines a tempered distribution.

If the test function ϕ vanishes in any small neighborhood of the origin, then $D^\alpha \phi(0) = 0$ for all α. Thus

$$\langle \Lambda_{m,N}, \phi \rangle = \int_{\mathbb{R}^d} \frac{\phi(x)}{|x|^m} \, dx.$$

holds, and $\Lambda_{m,N}$ coincides with $|x|^{-m}$ on $\mathbb{R}^d \setminus \{0\}$.

Discussion – Part B. Suppose such a tempered distribution $\Lambda \in \mathcal{S}'(\mathbb{R})$ exists. Then there is a $k \in \mathbb{N}$ with $|\langle \Lambda, \phi\rangle| \le C\|\phi\|_k$, where the norm $\|\phi\|_k$ is as in part A of this problem. If we only consider test functions that are supported in the interval $(0,1)$, then this would imply

$$\left|\int_0^1 e^{1/|x|}\phi(x)dx\right| \le C \sup_{0<x<1} |\phi^{(k)}(x)|.$$

We do not have to include the lower derivatives on the right, because for functions ϕ that are supported in an interval of length 1 these are bounded by the derivative of order k.

Now choose one such test function with $\phi(x) \ge 0$, and consider $\phi_\delta(x) = \phi(x/\delta)$, for small $\delta > 0$. Then

$$\int_0^1 e^{1/x}\phi_\delta(x)dx \le C \sup_{0<x<1} |\phi_\delta^{(k)}(x)|$$

for all $\delta \in (0,1)$. On the left we get

$$\int_0^1 e^{1/x}\phi_\delta(x)dx \overset{x=\delta t}{=} \delta\int_0^1 e^{1/(\delta t)}\phi(t)dt \ge \delta e^{1/\delta}\int_0^1 \phi(t)dt = C\delta e^{1/\delta}.$$

On the right we get

$$C \sup_{0<x<1} |\phi_\delta^{(k)}(x)| = C\delta^{-k} \sup_{0<t<1} |\phi^{(k)}(t)| = C'\delta^{-k}.$$

The assumption that a distribution Λ exists that restricts to $e^{1/x}$ on $\mathbb{R}_+$ therefore leads us to the conclusion that

$$C\delta e^{1/\delta} \le C'\delta^{-k}, \text{ i.e. } \delta^{k+1}e^{1/\delta} \le C''$$

for some constant C'' and for all $\delta \in (0,1)$. This is a contradiction.

Problem 8. Show that there is a continuous real valued function on [0,1] that is not monotone on any open interval $(a,b) \subset [0,1]$.

Discussion. There are (at least) two possible approaches: one can use an abstract Baire category argument, or one can try to construct an explicit example of such a function.

Baire category approach. Let $V_n \subset C([0,1])$ be the set of functions f such that for all $k = 1,\dots,2^n-1$ one has

$$f\Big(\frac{2k}{2^n}\Big) > f\Big(\frac{2k+1}{2^n}\Big) \text{ and } f\Big(\frac{2k}{2^n}\Big) > f\Big(\frac{2k-1}{2^n}\Big)$$

Thus $f \in V_n$ implies that f is not monotone on any interval $\big((2k-1)2^{-n}, (2k+1)2^{-n}\big)$.

Clearly V_n is an open subset of $C([0,1])$.

While V_n itself is not dense in $C([0,1])$ we claim that $W_n = \cup_{k\ge n} V_k$ is dense. Once we have proved that, Baire's theorem implies that $\cap_{n=1}^\infty W_n$ is still dense in $C([0,1])$, and thus not empty. There is therefore some function f that lies in infinitely many V_n, and which is therefore not monotone on any open interval in $[0,1]$.

To show that W_n is in fact dense, we consider any $f \in C([0,1])$, and any $\epsilon > 0$. Since f is uniformly continuous we can find an integer $N \ge n$ such that

$$\forall x, y : |x-y| \le 2^{1-N} \implies |f(x) - f(y)| \le \epsilon$$

Consider

$$g(x) = f(x) + \epsilon\cos(2^N\pi x)$$

Then, writing $x_k = k2^{-N}$, we have

$$g(x_{2k}) - g(x_{2k\pm1}) = f(x_{2k}) - f(x_{2k\pm1}) + 2\epsilon \ge \epsilon > 0.$$

It follows that $g \in V_N \subset W_n$, while $\|f - g\|_\infty = \epsilon$, thereby proving that W_n is indeed dense in $C([0,1])$.

An explicit construction. Instead of using the Baire category theorem, one could also try to present a concrete example of a nowhere monotone function f.

A monotone continuous function is almost everywhere differentiable, so to find a nowhere monotone function one could look for nowhere differentiable functions. There are several types of examples of such functions. For example, a typical path of Brownian motion is nowhere differentiable (that example would require one to first construct Brownian motion). The most classical example of a nowhere differentiable continuous function is the one given by Weierstrass:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(\pi b^n x).$$

where $0 < a < 1$ and $ab > 1$. Under these hypotheses Weierstrass' function is known to be nowhere differentiable, but a detailed proof is not simple. We are asked to provided only one example, so we allow ourselves to make a few extra assumptions on a and b along the way. We prove that the function f above is nowhere monotone, at least if we assume $b \in \mathbb{N}$ is even and sufficiently large.

The guiding idea behind the following arguments is that at any length scale there is one term in the series for f (the one with $n = N$ for some suitably chosen N) that oscillates the most: all previous terms ($n < N$) do not oscillate much because their derivative turns out to be small, and all subsequent terms ($n > N$) are periodic with a very small period, so as long as we only evaluate the function at points separated by multiples of this very small period, the terms with $n > N$ will actually be constant.

Write f as

$$f(x) = \sum_{n<N} a^n \cos(\pi b^n x) + a^N \cos(\pi b^N x) + \sum_{n>N} a^n \cos(\pi b^n x) = P(x) + Q(x) + R(x).$$

We have

$$|P'(x)| \le \sum_{n<N} \pi (ab)^n < \frac{\pi (ab)^N}{ab - 1}.$$

The tail sum $R(x)$ is periodic with period $\frac{1}{2} b^{-N-1}$.

If $I \subset [0,1]$ is any open interval, and N is any sufficently large integer, then we can find k such that $(2k \pm 1) b^{-N} \in I$.

Periodicity of $R(x)$ implies that

$$R\big(2kb^{-N}\big) = R\Big((2k-1)b^{-N}\Big) = R\Big((2k+1)b^{-N}\Big).$$

The derivative bound for $P(x)$ implies

$$\Big|P\big((2k \pm 1)b^{-N}\big) - P\big(2kb^{-N}\big)\Big| \le \sup_{x\in[0,1]} |P'(x)|\, b^{-N} \le \frac{\pi a^N}{ab-1}.$$

The middle term $Q(x)$ satisfies

$$Q\big((2k+1)b^{-N}\big) - Q\big(2kb^{-N}\big) = Q\big((2k-1)b^{-N}\big) - Q\big(2kb^{-N}\big) = -2a^N.$$

Combining P, Q, R we get

$$f\big((2k \pm 1)b^{-N}\big) - f\big(2kb^{-N}\big) \le -2a^N + \frac{\pi}{ab-1} a^N.$$

If we choose b so large that $\pi/(ab-1) < 2$, then

$$f\big((2k \pm 1)b^{-N}\big) - f\big(2kb^{-N}\big) < 0$$

so that f is not be monotone on the interval $((2k-1)b^{-N}, (2k+1)b^{-N})$.

Problem 9. Let $f_n \in L^2(\mathbb{R})$ be a sequence of functions with $\|f_n\|_2 \le 1$ for all $n \in \mathbb{N}$. Assume that for each $\epsilon > 0$, there exists $R_\epsilon > 0$ so that

$$\forall n \in \mathbb{N} \quad \int\limits_{|x|>R} |f_n(x)|^2 dx + \int\limits_{|\xi|>R_\epsilon} |\hat{f}_n(\xi)|^2 d\xi < \epsilon$$

Prove that $\{f_n\}$ has a subsequence that converges in the L^2 norm.

Discussion. Let ϕ be a Schwartz function that is supported in $\{|x| \le 1\}$ and has $\phi(0) = \hat{\phi}(0) = 1$, and define $\phi_N(x) = N\phi(Nx)$, $N \in \mathbb{N}$.

Let N momentarily be fixed, and consider the functions $f_n^N(x) := \phi(\frac{x}{N})(f_n * \phi_N(x))$. Multiplication by a Schwartz function maps H^1 into itself, so

$$\|f_n^N\|_{H^1(\mathbb{R})} \le C_{\phi_N} \|f_n * \phi_N\|_{H^1(\mathbb{R})}.$$

The definition of $H^1(\mathbb{R})$ and the formula $\widehat{f * g} = \hat{f}\hat{g}$, then Hölder's inequality, then rapid decay of $\hat{\phi}_N$ and L^2-boundedness of the sequence f_n give us

$$\|f_n * \phi_N\|_{H^1(\mathbb{R})} = \|\langle\xi\rangle \hat{\phi}_N \hat{f}_n\|_{L^2} \le \|\langle\xi\rangle \hat{\phi}_N\|_{L^\infty} \|f_n\|_{L^2} \le C_{\phi_N};$$

here $\langle\xi\rangle := \sqrt{1+|\xi|^2}$. In summary, f_n^N is a bounded sequence in H^1 whose support is contained in $\{|x| \le N\}$.

By Rellich, for each N, there exists a subsequence $\{f_{n_j(N)}\}$ such that $\{f^N_{n_j(N)}\}$ converges in L^2. By proceeding iteratively, we may assume that $\{f_{n_j(N)}\}$ is a subsequence of $\{f_{n_j(N')}\}$ for each $N' \le N$. Let $n_j := n_j(j)$. Then for each N, $\{f^N_{n_j}\}$ converges in L^2.

We claim that $\{f_{n_j}\}$ converges in L^2. By completeness of L^2, it suffices to prove that $\{f_{n_j}\}$ is Cauchy. By the triangle inequality,

$$\|f_{n_j} - f_{n_k}\|_{L^2} \le \|f^N_{n_j} - f^N_{n_k}\|_2 + \|f^N_{n_j} - f_{n_j}\|_2 + \|f^N_{n_k} - f_{n_k}\|_2.$$

As $\{f^N_{n_j}\}$ is Cauchy for each N, it suffices to prove that for all $\varepsilon > 0$, there exists $N = N_\varepsilon$ such that $\|f_n^N - f_n\|_2 < \varepsilon$, for all n. We compute

$$\begin{aligned}\|f_n^N - f_n\|_2 &\le \|\phi(\tfrac{x}{N})(f_n * \phi_n - f_n)\|_2 + \|(1-\phi(\tfrac{x}{N}))f_n\|_2 \\ &\le \|f_n * \phi_n - f_n\|_2 + \|(1-\phi(\tfrac{x}{N}))f_n\|_2 = \|(1-\hat{\phi}(\tfrac{\xi}{N}))\hat{f}_n\|_2 + \|(1-\phi(\tfrac{x}{N}))f_n\|_2.\end{aligned}$$

Finally,

$$\begin{aligned}\|(1-\phi(\tfrac{x}{N}))f_n\|_2 &\le \|(1-\phi(\tfrac{x}{N}))f_n\|_{L^2(\{|x|<c_\varepsilon N\})} + \|(1-\phi(\tfrac{x}{N}))f_n\|_{L^2(\{|x|>c_\varepsilon N\})} \\ &\le \|1-\phi(x)\|_{L^\infty(\{|x|<c_\varepsilon\})}\|f_n\|_{L^2} + \|1-\phi\|_{L^\infty}\|f_n\|_{L^2(\{|x|>c_\varepsilon N\})}.\end{aligned}$$

For c_ε sufficiently small, the first term is bounded by ε, by continuity and $\phi(0) = 1$; for $N = c_\varepsilon^{-1} R_\varepsilon$ (with R_ε as in the hypothesis), the second term is also bounded by ε. Smallness of $\|(1-\hat{\phi}(\frac{\xi}{N}))\hat{f}_n\|_2$ is proved in exactly the same way. This completes the proof that f_{n_j} is Cauchy in L^2, and hence that it is convergent.