QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics
University of Wisconsin-Madison
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Version for Math 725

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ denotes the unit disc in the complex plane.
(2) For points x and y in $\mathbb{R}^n$, $|x - y|$ denotes the Euclidean distance between the points.
(3) If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then $\big|E\big|$ denotes its Lebesgue measure.
(4) If μ is a positive measure on a set X, and if f is a complex valued measurable function on X, then for $1 \leq p < +\infty$,

$$||f||_p = \Big[\int_X |f(x)|^p \, d\mu(x)\Big]^{1/p}.$$

Two functions on X are said to be equivalent if they are equal except on a set of μ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $||f||_p < +\infty$.

(5) If μ is a positive measure on a set X, and if f is a complex valued measurable function on X, then

$$||f||_\infty = \inf\big\{t > 0 \,\big|\, \mu\big(\{x \in X \,\big|\, |f(x)| > t\}\big) = 0\big\}.$$

$L^\infty(X) = L^\infty(X, d\mu)$ is the space of equivalence classes of measurable, complex valued functions on X such that $||f||_\infty < +\infty$.

(6) $L^p(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ denote the spaces of equivalence classes of functions as defined in (5) and (6) where the measure $d\mu$ is Lebesgue measure.
(7) $L^p_{\text{loc}}(\mathbb{R}^n)$ is the space of equivalence classes of measurable, complex valued functions on $\mathbb{R}^n$ which belong to $L^p(K)$ for every compact set $K \subset\subset \mathbb{R}^n$.
(8) If f and g are measurable functions on $\mathbb{R}^n$, the convolution $f * g$ is defined to be the function

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - t)\, g(t)\, dt$$

whenever the integral converges. Here dt denotes Lebesgue measure.

(9) If T is a distribution and φ is a test function, then $\big\langle T, \varphi\big\rangle$ denotes the value of the distribution applied to the test function.
(10) A Hilbert space H is a complete separable vector space with the inner product denoted by $\langle \cdot, \cdot \rangle$.

The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.

Problem 1.

(a) Show that the series
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\left(\frac{x}{n}\right)$$
converges pointwise to some function f on $\mathbb{R}$.

(b) Is f continuous on $\mathbb{R}$? Does $f'(x)$ exist for each $x \in \mathbb{R}$?

(c) Does the series converge uniformly on $\mathbb{R}$?

Problem 2. Identify all $x \in \mathbb{R}$ such that
$$\lim_{n\to\infty} \sin(nmx)$$
exists for some positive integer m.

Problem 3. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers and assume that
$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 1.$$

(a) Show that $\lim_{n\to\infty} a_n n^{-1} = 0$.

(b) For $b_n = \max\{a_1, \ldots, a_n\}$, show that $\lim_{n\to\infty} b_n n^{-1} = 0$.

(c) Show that for $\beta \geq 0$,
$$\lim_{n\to\infty} \frac{a_1^\beta + \cdots + a_n^\beta}{n^\beta}$$
exists and equals 0 and ∞ when $\beta > 1$ and $\beta < 1$, respectively.

Problem 4. Let $E \subseteq \mathbb{R}$ be measurable and satisfy
$$E + r = E$$
for every rational number r. Show that either E or its complement has measure 0.

Problem 5. Find all $f \in L^2([0, \pi])$ such that
$$\int_0^\pi |f(x) - \sin x|^2 dx \leq \frac{4\pi}{9}$$
and
$$\int_0^\pi |f(x) - \cos x|^2 dx \leq \frac{\pi}{9}.$$

Problem 6. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f\colon X \to \mathbb{R}$ be measurable. Prove that if $1 \leq p < r < q < \infty$ and there is $C < \infty$ such that
$$\mu\left(\{x : |f(x)| > \lambda\}\right) \leq \frac{C}{\lambda^p + \lambda^q}$$
for every $\lambda > 0$, then $f \in L^r(\mu)$.

Problem 7. Let $p \in (1, \infty)$, and for $f \in L^p(\mathbb{R})$ define

$$Tf(x) = \int_0^1 f(x+y)\, dy.$$

(a) Show that $\|Tf\|_p \le \|f\|_p$, and equality holds if and only if $f = 0$ almost everywhere.
(b) Prove that the map $f \mapsto f - Tf$ does not map $L^p(\mathbb{R})$ onto $L^p(\mathbb{R})$.

Problem 8. Let $\chi \in C^\infty(\mathbb{R})$ have a compact support and define

$$f_n(x) = n^2 \chi'(nx)\,.$$

(a) Does f_n converge in the sense of distributions as $n \to \infty$? If so, what is the limit?
(b) Let $p \in [1, \infty)$ and $g \in L^p(\mathbb{R})$ be such that the distributional derivative of g also lies in $L^p(\mathbb{R})$. Does $f_n * g$ converge in $L^p(\mathbb{R})$ as $n \to \infty$? If so, what is the limit?

Problem 9. Recall that $H^s(\mathbb{R}^n)$ (with $s \ge 0$) is the Sobolev space consisting of all tempered distributions g on $\mathbb{R}^n$ for which the Fourier transform $\widehat{g}$ of g is locally integrable and satisfies

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{g}(\xi)|^2 d\xi < \infty.$$

Let u be a Schwartz function on $\mathbb{R}^n$ and for $a \in \mathbb{C}$, let

$$f_a(x) = |x|^a u(x).$$

Show that if $\operatorname{Re} a > -\frac{n}{2}$ and $s \in [0, \operatorname{Re} a + \frac{n}{2})$, then $f_a \in H^s(\mathbb{R}^n)$.