An Introduction to Constructive Mathematics

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Problematic Proofs

Theorem

There exist irrational numbers a, b such that a^b is rational.

Proof.

Suppose that $\sqrt{2}^{\sqrt{2}}$ is rational. Then we can simply take $a = b = \sqrt{2}$. But if $\sqrt{2}^{\sqrt{2}}$ is irrational, then we may put $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ and see:

$$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^2 = 2.$$

□

AoC $\Rightarrow$ LEM

Theorem (Diaconescu)

We can derive the Law of Excluded Middle from the Axiom of Choice.

Proof.

Let $X = \{0, 1\}$ and let p be a proposition. Define the following sets:
$A = \{x \in \{0, 1\} | (x = 1) \vee p\}$,
$B = \{x \in \{0, 1\} | (x = 0) \vee p\}$.
If p is true $A = B = \{0, 1\}$, and we have

$$p \to A = B$$

AoC $\Rightarrow$ LEM

Proof.

$$p \to f(A) = f(B) \text{ and } f(A) \neq f(B) \to \neg p$$

Then $1 \in A$ and $0 \in B$ so both A and B are nonempty. Let $\mathcal{X} = \{A, B\}$. By the Axiom of Choice: $f : \mathcal{X} \to X$ as $\bigcup \mathcal{X} = X$. We want $f(A) \in A$ and $f(B) \in B$

$$(f(A) = 1 \vee p) \wedge (f(B) = 0 \vee p)$$

$$(f(A) \neq f(B)) \vee p$$

$$p \vee \neg p$$

. □

Brouver-Heyting-Kolmogoroff (BHK) Interpretation

Conjunction: A proof $\varphi \wedge \psi$ is a pair $\langle p, q \rangle$ where p is a proof of φ and q is a proof of ψ.

Implication: A proof of $\varphi \to \psi$ is a (constructive) function f mapping proofs of φ to proofs of ψ.

Falsity: There is no proof of $\bot$.

Disjunction: A proof p of $\varphi \vee \psi$ is either a proof of φ or a proof of ψ where is it indicated whether p proves φ or ψ.

Brouver-Heyting-Kolmogoroff (BHK) Interpretation

Universal Quantification: A proof of $\forall x.\varphi(x)$ is a (constructive) function f such that $f(d)$ is a proof of $\varphi(d)$ for all $d \in D$ where D is the domain over which the variable x ranges.

Existential Quantification: A proof of $\exists x.\varphi(x)$ is a pair $\langle d, p\rangle$ where $d \in D$ and p is a proof of $\varphi(d)$ where D is the domain over which x ranges.

Review of Classical Rules

Some axioms of classical predicate logic are:

1. $\varphi \to (\psi \to \varphi)$ for any formulas φ, ψ
2. $\neg\neg\varphi \to \varphi$ for any formula φ
3. $(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to \theta)$ for any formulas φ, ψ, θ

The deduction rules:

1. Modus Ponens: From φ and $\varphi \to \psi$, we can deduce ψ
2. Generalization: From φ, we can deduce $\forall x.\varphi(x)$

Review of Classical Rules

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Natural Deduction Rules

$\wedge$ Elimination:

$$\frac{\Omega \vdash \varphi_1 \wedge \varphi_2}{\Omega \vdash \varphi_i} \; (\wedge E_i)$$

$\vee$ Introduction:

$$\frac{\Omega \vdash \varphi_i}{\Omega \vdash \varphi_1 \vee \varphi_2} \; (\vee I_i)$$

Axiom:

$$\frac{}{\Omega, \varphi, \Delta \vdash \varphi} \; (\text{ax})$$

$\rightarrow$ Introduction:

$$\frac{\Omega, \varphi \vdash \psi}{\Omega \vdash \varphi \rightarrow \psi} \; (\rightarrow I)$$

$\rightarrow$ Elimination:

$$\frac{\Omega \vdash \varphi \rightarrow \psi \qquad \Omega \vdash \varphi}{\Omega \vdash \psi} \; (\rightarrow E)$$

$\perp$ Elimination:

$$\frac{\Omega \vdash \perp}{\Omega \vdash \theta} \; (\perp E)$$

$\neg$ vs. $\bot$

In constructive logic, rather than have $\neg$ be a foundational propositional connective, we take $\bot$ as a constant and define the shorthand:

$$\neg\varphi := \varphi \to \bot$$

LEM→RAA

$\Omega := \varphi \vee \neg\varphi, \neg\neg\varphi$:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\Omega, \neg\varphi \vdash \neg\neg\varphi}\;(\text{ax})
      \qquad
      \dfrac{}{\Omega, \neg\varphi \vdash \neg\varphi}\;(\text{ax})
    }{\Omega, \neg\varphi \vdash \bot}\;(\to E)
  }{\Omega, \neg\varphi \vdash \varphi}\;(\bot E)
  \qquad
  \dfrac{}{\Omega \vdash \varphi \vee \neg\varphi}\;(\text{ax})
  \qquad
  \dfrac{}{\Omega, \varphi \vdash \varphi}\;(\text{ax})
}{
  \dfrac{\varphi \vee \neg\varphi \neg\neg\varphi \vdash \varphi}{\varphi \vee \neg\varphi \vdash \neg\neg\varphi \to \varphi}\;(\to I)
}\;(\vee E)
$$

Truth-Value Semantics

Definition

A lattice is a partially ordered set X in which every finite subset has a least upper bound and a greatest lower bound.
For $a, b \in X$ we write $a \vee b$ ('a join b') for the supremum $\{a, b\}$ and $a \wedge b$ ('a meet b') for the infimum of $\{a, b\}$. These operations are commutative and associative.

Truth-Value Semantics

Definition

We say a lattice X is a <u>Boolean Algebra</u> if it contains a greatest element $\top$ and least element $\bot$ and if it has the following properties:

1. For all $a, b, c \in X: \; a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
2. For all $a, b, c \in X: \; a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.
3. For every $a \in X$ there exists $\neg a \in X$ such that $(a \vee \neg a = 1)$ and $(a \wedge \neg a = 0)$.

Truth-Value Semantics

Definition

A Heyting algebra is a partially ordered set $(H, \leq)$ with meets and joins for every finite subset such that for all $a, b \in H$ there exists $a \to b \in H$ with

$$c \leq a \to b \quad \text{if and only if} \quad c \wedge a \leq b.$$

$\to$ is a binary operation on H called Heyting implication.

Heyting algebras are always distributive!

Truth-Value Semantics

Theorem

If there is a derivation of $\varphi_1, \dots, \varphi_n \vdash \psi$ in the propositional part of Natural Deduction, then there exists a Heyting Algebra H and a valuation ρ such that

$$\rho(\varphi) \wedge_H \cdots \wedge_H \rho(\varphi_n) \leq_H \rho(\psi).$$

Proof.

Easy, but long and detail-rich induction proof. □

Constructively Invalidating Classical Tautologies

Theorem

In Constructive Predicate Calculus, one cannot derive

$$\neg\neg p \to p; \qquad p \vee \neg p$$

for propositional constants p and q.

Constructively Invalidating Classical Tautologies

Let P be a partially ordered set and consider
$\mathrm{dcl}(P) \coloneqq \{A \in \mathcal{P}(P) \ : \ y \leq x \in A \to y \in A\}$

We define $\downarrow x \coloneqq \{y \in P \ : \ y \leq x\}$

We then have Heyting implication given by:
$U \to V = \bigcup \{\downarrow x \in P \ : \ U \cap \downarrow x \subseteq V\}$

Equivalently: $U \to V \coloneqq \{x \in P \ : \ \forall y \leq x.\ y \in U \to y \in V\}$

Constructively Invalidating Classical Tautologies

Let H be the Heyting algebra $\mathrm{dcl}(\mathbf{2})$ where $\mathbf{2}$ is the partially ordered set $0 < 1$. Let $u = \downarrow 0$ i.e. $u = \{0\}$.

$$\neg u = u \to \bot = \{0\} \to \emptyset = \bigcup \{\downarrow x \in \{0,1\} : \{0\} \cap \downarrow x \subseteq \emptyset\}$$

$$= \emptyset = \bot.$$

$$\neg\neg u = \bigcup \{\downarrow x \in \{0,1\} \ : \ \emptyset \cap \downarrow x \subseteq \emptyset\} = \{0,1\} = \top.$$

A Boolean Algebra...

Definition

A Boolean algebra is a Heyting algebra B such that $\neg\neg a \leq a$ for all $a \in B$.

Power of Constructivism

Definition

Gödel-Gentzen double negation translation $(-)^G$:

- $(\varphi \vee \psi)^G := \neg(\neg\varphi^G \wedge \neg\psi^G)$.
- $(\exists x.\varphi)^G := \neg\forall x.\neg\varphi^G$.

Theorem

Let T be a theory, such that for every $\varphi \in T$, the formula φ^G is constructively derivable from T. Then φ is classically derivable from T if and only if φ^G is constructively derivable from T.

Conclusion

"Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists. To prohibit existence statements and the principle of excluded middle is tantamount to relinquishing the science of mathematics altogether."- David Hilbert

Sources

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