

ALGEBRA QUALIFYING EXAM, JANUARY 2019

On the first problem, only the answer will be graded. On all other problems, you will be expected to justify all responses. Each problem is worth 20 points. Unless otherwise stated, parts of a given problem will be worth roughly the same amount.

- (1) On this problem, only the answers will be graded.
 - (a) Give an example of a simple ring which is not a commutative ring.
 - (b) Give an example a ring R , R -modules M_1, M_2, M_3 , and N and a short exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ such that the corresponding sequence $0 \leftarrow \text{Hom}(M_1, N) \leftarrow \text{Hom}(M_2, N) \leftarrow \text{Hom}(M_3, N) \leftarrow 0$ is not exact.
 - (c) Let Φ be the map $\mathbb{Z}^3 \rightarrow \mathbb{Z}^2$ given by the matrix $\begin{bmatrix} 2 & 3 & 5 \\ 12 & -4 & 8 \end{bmatrix}$. Let M be the cokernel of Φ , which we recall is $\mathbb{Z}^2/\text{image}(\Phi)$. Compute the annihilator of M .

- (2) If G is a group and $x, y \in G$, then we let $\langle x, y \rangle$ denote the subgroup of G generated by x and y . For $x \in G$, define $\text{Cyc}_G(x) := \{y \in G \mid \langle x, y \rangle \text{ is cyclic}\}$.
 - (a) Show by example that $\text{Cyc}_G(x)$ need not be a subgroup of G .
 - (b) Let $\text{Cyc}(G) = \bigcap_{x \in G} \text{Cyc}_G(x)$. Show that $\text{Cyc}(G)$ is a subgroup of G .
 - (c) Show that $\text{Cyc}(G)$ lies in the center of G .
 - (d) Assume now that G is finite. Show that $\text{Cyc}(G)$ is cyclic.

- (3) Let R be a commutative ring and let $S \subseteq R$ be a multiplicatively closed subset.
 - (a) (12 points) Let M be an R -module. Prove that $S^{-1}M \cong S^{-1}R \otimes_R M$.
 - (b) (8 points) Give an example of a ring R , a multiplicatively closed subset $S \subseteq R$ and an R -module M where $S^{-1}M$ is flat and nonzero, but where M is not flat.

- (4)
 - (a) Let E be the splitting field over \mathbb{Q} of the polynomial $x^3 + 2x^2 + 3x + 4$ (which has discriminant -200). Find $\text{Gal}(E/\mathbb{Q})$.
 - (b) Let F be the splitting field over \mathbb{Q} of the polynomial $x^3 + 4x^2 + 7x + 6$ (which has discriminant -72). Find $\text{Gal}(F/\mathbb{Q})$.
 - (c) Let E and F be as above. Find $\text{Gal}(EF/\mathbb{Q})$.

- (5) Let F be a field and n be a positive integer. Fix an $n \times n$ matrix S over F that is invertible and symmetric. Writing A^t for the transpose of a matrix, we let
$$V := \{n \times n \text{ matrices } A \text{ over } F \mid A^t = SAS^{-1}\}$$
Note that V is a vector space (you do not need to prove this). Find the dimension of V in terms of n .

SOLUTIONS TO JANUARY 2019 ALGEBRA QUALIFYING EXAM

- (1) (a) Any non-commutative division ring will work.
 (b) $R = \mathbb{Z}$ and $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ and $N = \mathbb{Z}/2$ will work.
 (c) Via row and column operations, the matrix is equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 44 & 0 \end{bmatrix}$ so the annihilator is (44).
- (2) If $x_1, x_2, \dots, x_n \in G$ then we will write $\langle x_1, x_2, \dots, x_n \rangle$ for the subgroup of G generated by x_1, x_2, \dots, x_n .
 (a) Let $G = \mathbb{Z}/2 \times \mathbb{Z}/4$ and $x = (0, 2)$. Then $(0, 1)$ and $(1, 1)$ are both in $\text{Cyc}_G(x)$: this is because $(0, 1) + (0, 1) = (0, 2) = (1, 1) + (1, 1)$ and so $\langle (0, 2), (0, 1) \rangle = \langle (0, 1) \rangle$ and similarly for $(1, 1)$. But $(1, 0) = (0, 1) + (1, 1)$ is not in $\text{Cyc}_G(x)$, since $\langle (0, 2), (1, 0) \rangle$ is a group of order 4 with no element of order 4, and is thus not cyclic.
 (b) If $g \in \text{Cyc}(G)$ and $x \in G$, then $\langle g^{-1}, x \rangle = \langle g, x \rangle$, which is cyclic, so $g^{-1} \in \text{Cyc}(G)$. If also $h \in \text{Cyc}(G)$, then $\langle gh, x \rangle$ is a subgroup of $\langle g, h, x \rangle$. But we claim that $\langle g, h, x, \rangle$ is also cyclic: this is because $\langle h, x \rangle$ is cyclic (since $h \in \text{Cyc}(G)$) and thus $\langle h, x \rangle = \langle y \rangle$ for some y . It follows that $\langle g, h, x \rangle = \langle g, y \rangle$ which is cyclic because $g \in \text{Cyc}(G)$. Therefore, $\langle gh, x \rangle$ is a subgroup of a cyclic group. So it must also be cyclic, which shows that $gh \in \text{Cyc}(G)$.
 (c) Let $g \in \text{Cyc}(G)$ and let $x \in G$. Since $\langle g, x \rangle$ is a cyclic, g and x must commute and thus g lies in the center of G .
 (d) Let H be any cyclic subgroup of $\text{Cyc}(G)$ which is maximal under inclusion. (Such a group exists since G is finite.) Then $H = \langle h \rangle$ for some h . For any $g \in \text{Cyc}(G)$ we would have that $\langle g, h \rangle$ is a cyclic subgroup of $\text{Cyc}(G)$ containing H and thus $\langle g, h \rangle = H = \langle h \rangle$. It follows that $H = \text{Cyc}(G)$.
- (3) Let R be a commutative ring and let $S \subseteq R$ be a multiplicatively closed subset.
 (a) Consider the map $S^{-1}R \times M \rightarrow S^{-1}M$ given by $(\frac{r}{s}, m) \mapsto \frac{rm}{s}$. We first check that the map is R -bilinear:
 • If we fix m then we must check that the map $\phi_m : S^{-1}R \rightarrow S^{-1}M$ sending $\frac{r}{s} \mapsto \frac{rm}{s}$ is R -linear. This follows since $\phi_m(\frac{r}{s}) + \phi_m(\frac{r'}{s'}) = \frac{rm}{s} + \frac{r'm}{s'} = \frac{rs'm + r'sm}{ss'} = \phi_m(\frac{rs'+r's}{ss'}) = \phi_m(\frac{r}{s} + \frac{r'}{s'})$ and $r'\phi_m(\frac{r}{s}) = r' \cdot \frac{rm}{s} = \frac{r'r'm}{s} = \phi_m(\frac{r'r}{s})$.
 • If we fix $\frac{r}{s}$ and let $\psi_{r/s} : M \rightarrow S^{-1}M$ be given by sending $m \mapsto \frac{rm}{s}$ then $\psi_{r/s}(m + m') = \frac{r(m+m')}{s} = \frac{rm}{s} + \frac{rm'}{s} = \psi_{r/s}(m) + \psi_{r/s}(m')$. And $r'\psi_{r/s}(m) = r' \cdot \frac{rm}{s} = \frac{r'r'm}{s} = \psi_{r/s}(r'm)$.
- Thus by the universal property of tensor products, there is a map of R -modules $S^{-1}R \otimes_R M \rightarrow S^{-1}M$ generated by sending $\frac{r}{s} \otimes m \mapsto \frac{rm}{s}$. The element $\frac{1}{s} \otimes m$ maps to $\frac{m}{s}$, so the map is surjective.
 We must also check injectivity. Consider the element $\sum_{i=1}^n \frac{a_i}{s_i} \otimes m_i$ in $S^{-1}R \otimes_R M$. Write $s = \prod_{i=1}^n s_i$ for the common denominator and $t_i = \prod_{j \neq i} s_j$. Then for each i we have $\frac{a_i}{s_i} = \frac{a_i t_i}{s}$. Now we can rewrite $\sum_{i=1}^n \frac{a_i}{s_i} \otimes m_i = \sum_{i=1}^n \frac{a_i t_i}{s} \otimes m_i$. By

properties of tensor products, we can rewrite this as $\sum_{i=1}^n \frac{1}{s} \otimes a_i t_i m_i$ which is in turn equal to $\frac{1}{s} \otimes \sum_{i=1}^n a_i t_i m_i$. Thus every element in $S^{-1}R \otimes_R M$ can be written in the form $\frac{1}{s} \otimes m$ for some $s \in S$ and some $m \in M$. So we now assume that some elements $\frac{1}{s} \otimes m$ maps to zero in $S^{-1}M$. This means that $\frac{m}{s} = 0$ in $S^{-1}M$ which means that there exists $t \in S$ such that $mt = 0$. Then we have

$$\frac{1}{s} \otimes m = \frac{t}{ts} \otimes m = \frac{1}{ts} \otimes tm = \frac{1}{ts} \otimes 0 = 0.$$

Thus the map is injective.

- (b) $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}/2$ and $S = \{2, 2^2, 2^3, \dots\}$. Works. First we show that M is not flat. A direct sum of modules is flat if and only if each summand is flat, but $\mathbb{Z}/2$ is not flat since tensoring $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ with $\mathbb{Z}/2$ will not preserve exactness. However, $S^{-1}R = \mathbb{Z}[2^{-1}]$ and $S^{-1}M = \mathbb{Z}[2^{-1}]$ and thus $S^{-1}M$ is a free $S^{-1}R$ -module, and hence it is flat.

- (4) (a) First we show the polynomial is irreducible. Since it is cubic, if it has a non-trivial factor, it has a linear factor. it has a non-trivial factor, it has a linear factor. By Gauss's theorem, it then has a factorization over the integers and thus an integral root. We check mod 3, that $x = 0$ gives a value of 1, and $x = 1$ gives a value of 1, and $x = 2$ gives a value of 2, and thus the equation has no roots mod 3 and thus no integral roots. So the cubic polynomial is irreducible and thus has Galois group either C_3 or S_3 . Since -200 is not a square, the Galois group is S_3 .
- (b) We factor $x^3 + 4x^2 + 7x + 6 = (x^2 + 2x + 3)(x + 2)$. So F is the splitting field of $x^2 + 2x + 3$, which is $\mathbb{Q}(\sqrt{-8})$, as $2^2 - 4 \cdot 3 = -8$. Every quadratic extension of \mathbb{Q} has Galois group C_2 over \mathbb{Q} , and so the Galois group is C_2 .
- (c) Note that $F = \mathbb{Q}(\sqrt{-2})$, and thus F is a subfield of E , since E contains the square root of the discriminant of $x^3 + 2x^2 + 3x + 4$, i.e. contains $\sqrt{-200}$ and hence contains $\sqrt{-2}$. Thus, $EF = E$, and the Galois group is S_3 .

- (5) We first claim that:

$$A \in V \iff SA \text{ is symmetric.}$$

This follows because $SAS^{-1} = A^t \iff SA = A^tS$. But S is symmetric so $A^tS = A^tS^t = (SA)^t$. Thus $A \in V$ if and only if $SA = (SA)^t$ or equivalently, $A \in V$ if and only if SA is symmetric. Let W be the vector space of $n \times n$ symmetric matrices, which has dimension $\binom{n+1}{2}$. Consider the linear transformation $\Phi : W \rightarrow V$ given by $M \mapsto SM$. Since S is invertible, the map $V \rightarrow W$ given by $N \mapsto S^{-1}N$ is an inverse of Φ and thus Φ is an isomorphism of vector spaces. It follows that $\dim V = \dim W = \binom{n+1}{2}$.