ALGEBRA QUALIFYING EXAM, FALL 2019

On the first problem, only the answer will be graded. On all other problems, you will be expected to justify all responses. Each problem is worth 20 points. Unless otherwise stated, parts of a given problem will be worth roughly the same amount.

(1) **On this problem, only the answers will be graded.**
   (a) Let $R = M_2(\mathbb{Z})$ the ring of the 2×2 matrices with entries in $\mathbb{Z}$. Give an example of left ideal that is not a right ideal.
   (b) Give an example in $R = M_2(\mathbb{Z})$ of a proper, nonzero, two-sided ideal.
   (c) $\mathbb{Z}/100\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/35\mathbb{Z}$ is a finite abelian group. Compute its order.

(2) This problem involves ideals in a commutative ring (with unit). A proper ideal I is **primary** if whenever $ab \in I$ and $a \notin I$ then $b^n \in I$ for some $n > 0$. The radical of an ideal I, denoted $\sqrt{I}$, is the set $\sqrt{I} = \{f | f^n \in I \text{ for some } n > 0\}$.
   (a) Prove that if I is primary then $\sqrt{I}$ is prime.
   (b) Is the ideal $J = (x^2, xy) \subseteq \mathbb{Q}[x, y]$ primary? Be sure to carefully justify your answer.

(3) Let L/K be a Galois extension of fields with Galois group G. Let S be the subset of roots of unity in L; that is, S is the set of elements $x \in L$ such that $x^k = 1$ for some positive integer k.
   (a) Show that the action of G on L restricts to an action of G on S (i.e., the action of G preserves S.)
   (b) Let $L = \mathbb{Q}(i)$ and $K = \mathbb{Q}$. Show in this case that G acts faithfully on S.
   (c) Give an example of L/K such that the action of G on S is not faithful.
   (d) Suppose that the action of G on S is transitive. Prove that the characteristic of K must be 2.

(4) This problem involves 4×4 matrices with entries in $\mathbb{R}$. We will say that a matrix M is an E-**matrix** if M has 1's along the diagonal and a single nonzero entry off of the diagonal. For instance:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 31 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is an E-matrix. Express the matrix A below as a product of E-matrices.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

(5) Let G be the group of 2×2 upper triangular matrices over the field $\mathbf{F}_p$.
    (a) Prove that G has only one subgroup of order p, and that this subgroup is normal. (Hint: first show that an element of order p must have 1's on the diagonal.)
    (b) Prove that G is solvable by exhibiting a homomorphism f from G to an abelian group A such that the kernel of f is also abelian.
    (c) Prove that G is not nilpotent.

(1) (a) The set of matrices of the form $\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$ works.

   (b) The set of matrices whose entries are a multiple of p works.

   (c) $(\mathbb{Z}/25 \oplus \mathbb{Z}/4) \otimes (\mathbb{Z}/5 \otimes \mathbb{Z}/7) = \mathbb{Z}/25 \otimes \mathbb{Z}/5 = \mathbb{Z}/5$. So it has order 5.

(2) (a) Let $ab \in \sqrt{I}$ and $a \notin \sqrt{I}$. We must show that $b \in \sqrt{I}$. Since $ab \in \sqrt{I}$ we have $(ab)^n = a^n b^n \in I$ for some n. But since $a \notin \sqrt{I}$ we have $a^n \notin I$. By definition of primary, this implies that $(b^n)^m \in I$ for some m. Thus $b^{nm} \in I$ which implies that $b \in \sqrt{I}$ as desired.

   (b) No J is not primary. Since $xy \in J$ and $x \notin J$ (because J contains no linear forms), it suffices to show that $y^n \notin J$ for all n. However, everything in J is divisible by x and thus $y^n \notin J$ for all n, as desired.

(3) (a) The n'th roots of unity in L are precisely the elements in L that are roots of the equations $t^n - 1$. But $t^n - 1$ has coefficient in K and thus G permutes the roots of this equation.

   (b) Since $[L : K] = 2$, G is the cyclic group of order 2. Also $S = \{\pm 1, \pm i\}$. The nontrivial element of G permutes i and $-i$, and thus this action is faithful.

   (c) Let $L = \mathbb{Q}(\sqrt{2})$ and $K = \mathbb{Q}$. Then $S = \{\pm 1\}$ since those are the only roots of unity in $\mathbb{R}$ and $L \subseteq \mathbb{R}$. As in the previous example G is a cyclic group of order 2. But both elements of G fix S since $S \subseteq \mathbb{Q}$.

   (d) Imagine K has characteristic not equal to 2. Then -1 and 1 are distinct elements of $S \cap K$. But every element of G will fix both -1 and 1 since they lie in K and thus the action could not be transitive.

(4) Row-reduce A down to the identity matrix I, and interpret each reduction step as left-multiplication by an elementary matrix. This yields an equation expressing the fact that a product of elementary matrices times A is equal to I. Using this equation solve for A to get the answer, which is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(5) (a) If g is an element of G with diagonal entries a_1, a_2, one computes that the diagonal entries of g^p are a_1^p, a_2^p. So if g is an element of order p, its diagonal entries must have order p. But the diagonal entries of G must be invertible, so they lie in $\mathbf{F}_p^*$, a group of order $p-1$; since $p-1$ is prime to p, this means a_1 and a_2 are both 1. So g must lie in the subgroup H of G consisting of matrices which

are 1 on the diagonal. But this group has order p, so it is the unique subgroup of G of order p. What?s more, any conjugate of H also has order p, so must be equal to H; this is exactly to say that H is normal.

(b) The argument above implicitly makes use of the fact that the map $G \to \mathbf{F}_p^* \times \mathbf{F}_p^*$ sending g to its diagonal entries is a homomorphism. The target is abelian, and the kernel is the group H above, which is also abelian.

(c) A finite nilpotent group is the direct sum of its p-Sylow subgroups; in particular, if a p-Sylow subgroup is abelian, it is central, and one easily sees that H is not central. Alternate proof: the second term fo the lower central series is the commutator subgroup $[G, G]$, which one can check directly is H; but the third term is $[G, H]$, which is nontrivial since H is not central, and being a nontrivial subgroup of a cyclic group of prime order is in fact all of H; so the lower central series stabilizes at H, not 1, and so G is not nilpotent.