Sections 3.4-3.6: Cohomology of line bundles

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1 Previous facts

The Dolbeault complex computes cohomology of a holomorphic vector bundle $\mathcal{E}$: it is

$$A^0(E) \xrightarrow{\overline{\partial}} A^1(E) \xrightarrow{\overline{\partial}} A^2(E) \longrightarrow \cdots$$

Where $A^p(E)$ is the sheaf of ($\mathcal{C}^\infty$-) antiholomorphic differentials. This relies on two facts:

1. The $\overline{\partial}$-Poincaré lemma ensures that the cohomology of this complex is just the holomorphic sections of E on degree 0.
2. All the sheaves $A^p(E)$ have no higher cohomology, because they're soft (both facts can be found as Proposition 2.18 in Dimca's Sheaves in Topology).

1.1 Computing cohomology using some linear algebra (i.e. Hodge theory: the easy parts)

Say we have some complex $(A^\bullet, d)$ of $\mathbb{C}$-vector spaces A^i, together with a positive definite hermitian form on each A^i. Suppose also there is an adjoint d^* to d (in the finite dimensional case this is the conjugate transpose matrix, but in general it might not exist, though it does if the spaces are complete, by the Riesz representation theorem). Then the hope is the following:

$$\ker d = \text{im } d \oplus (\ker d \cap \ker d^*)$$

This is not always true, but when it is our life is easier.

We can try to prove it as follows:

1. The intersection is 0: let $da \in \ker d^*$, i.e. $d^* da = 0$. Then
$$\|da\|^2 = (da, da) = (d^* da, a) = (0, a) = 0$$
Which implies that $da = 0$.
2. $A = \text{im } d + \ker d^*$. Let us show that $\text{im } d^\perp \subset \ker d^*$. Let a be in the left hand side, i.e. for any b, $(a, db) = 0$. Then for any b,
$$(d^* a, b) = (a, db) = 0$$
Which implies that $d^* a = 0$. Now we have that
$$(\text{im } d + \ker d^*)^\perp = \text{im } d^\perp \cap (\ker d^*)^\perp \subset \ker d^* \cap (\ker d^*)^\perp = 0$$
Does this imply that $\text{im } d + \ker d^* \supset \ker d$? It would if A was complete, but in general it implies that
$$\overline{\text{im } d + \ker d^*} \supset \ker d$$
Which is not good enough! Anyway, the fact that we can go around this is contained in the Hodge theorem, in the case where $A^\bullet$ is the Dolbeault complex.

Knowing this, we have that $H^\bullet(A^\bullet) \cong \ker d \cap \ker d^*$. Finally, we can easily see that

$$\ker d \cap \ker d^* = \ker dd^* + d^* d$$

The inclusion from left to right is clear. If a is in the right hand side,

$$0 = (dd^* a + d^* da, a) = (d^* a, d^* a) + (da, da) = \|da\|^2 + \|d^* a\|^2$$

So we have that the cohomology of the complex is isomorphic to the harmonic stuff in it, i.e. $\ker d^* d + dd^*$.

2 "A vanishing theorem"

Let $X = V/L$ an abelian variety.

Remember that the Appel-Humbert data is H a Hermitian form whose imaginary part is integer-valued on L, and $\alpha : L \longrightarrow U(1)$ such that $\alpha(l_1 + l_2) = \alpha(l_1)\alpha(l_2)(-1)^{\Im H(l_1,l_2)}$.

A ($\mathcal{C}^\infty$, holomorphic) section of (α, H) is a ($\mathcal{C}^\infty$, holomorphic) function f on V, that is L-equivariant in the following way:

$$f(v+l) = \alpha_l e^{\pi H(v,l) + \frac{\pi}{2} H(l,l)} f(v)$$

Computation. *On the sections of (α, H), the following inner product is well-defined:*

$$(f,g) = \int_X e^{-\pi H(v,v)} f(v)\overline{g(v)} dv$$

Proof. Let us show that $e^{-\pi H(v,v)} f(v)\overline{g(v)}$ is L-invariant:

$$\begin{aligned} e^{-\pi H(v+l,v+l)} f(v+l)\overline{g(v+l)} &= e^{-\pi H(v+l,v+l)} f(v)\overline{g(v)}\alpha_l\overline{\alpha_l} e^{\pi H(v,l)+\frac{\pi}{2}H(l,l)} e^{\pi\overline{H(v,l)}+\frac{\pi}{2}H(l,l)} = \\ &= f(v)\overline{g(v)} e^{-\pi H(v+l,v+l)+\pi H(v,l)+\pi H(l,v)+\pi H(l,l)} = \\ &= f(v)\overline{g(v)} e^{-\pi H(v,v)} \end{aligned}$$

We are happy. ∎

Now we are assuming H is diagonal (do a linear change of basis on V by the Gramm-Schmidt process), so $H(z,w) = \sum h^j z_j \overline{w_j}$, and we are taking $\frac{\partial}{\partial \overline{z_j}} : A^0 \to A^0$.

Since A^0 has this inner product, we can look at the adjoint to $\frac{\partial}{\partial \overline{z_j}}$. Turns out that

$$\left(\frac{\partial}{\partial \overline{z_j}}\right)^* = -\frac{\partial}{\partial z_j} + \pi h^j \overline{z}_j$$

Lemma (Lemma 3.6)**.** *1. $-\frac{\partial}{\partial z_j} + \pi h^j \overline{z}_j$ is well-defined, in that it preserves A^0*

2. They are indeed adjoints.

3. The commutator

$$\left[\frac{\partial}{\partial \overline{z_j}}, \frac{\partial}{\partial \overline{z_j}}^*\right] = \pi h^j$$

Proof. 1. Follow the computation of the book to show

$$\left(\left(-\frac{\partial}{\partial z_j} + \pi h^j \overline{z}_j\right) f\right)(z+l) = \alpha_l e^{\pi H(z,l)+\frac{\pi}{2}H(z,l)} \left(-\frac{\partial}{\partial z_j} + \pi h^j \overline{z}_j\right) f(z)$$

It amounts to showing that whatever shows up from the Leibniz rule when we take the derivative cancels with the extra term.

2. For the Stokes' theorem argument, we are trying to show that

$$\left(\frac{\partial}{\partial \overline{z_j}} f, g\right) - \left(f, \frac{\partial}{\partial \overline{z_j}}^* g\right) = \frac{\partial}{\partial \overline{z_j}} (f,g)_{pt}$$

If the identity holds, we have that

$$-\frac{\partial}{\partial \overline{z_j}} (f,g)_{pt} \partial z_1 \wedge \cdots \wedge \partial \overline{z}_1 \wedge \cdots = \pm d((f,g)_{pt} \partial z_1 \wedge \cdots \wedge \partial \overline{z}_1 \wedge \widehat{\partial \overline{z}_j} \wedge \cdots)$$

So the form on the left hand side is exact, let us just write it as $d\alpha$, therefore

$$\begin{aligned} \left(\frac{\partial}{\partial \overline{z_j}} f, g\right) - \left(f, \frac{\partial}{\partial \overline{z_j}}^* g\right) &= \int_X \left(\frac{\partial}{\partial \overline{z_j}} f, g\right)_{pt} - \left(f, \frac{\partial}{\partial \overline{z_j}}^* g\right)_{pt} dv = \\ &= \int_X \frac{\partial}{\partial \overline{z_j}} (f,g)_{pt} dv = \int_X d\alpha \overset{\text{(Stokes)}}{=} \int_{\partial X} \alpha = 0 \end{aligned}$$

To show the identity above, we do the following (THE COMPUTATION AT THE TOP OF PAGE 24 IS BS)

$$\frac{\partial}{\partial \overline{z}_j}(f,g)_{pt} = \frac{\partial}{\partial \overline{z}_j}\left(e^{-\pi H(z,z)} f(z)\overline{g(z)}\right)_{pt} =$$
$$= -\pi h^j \overline{z}_j e^{-\pi H(z,z)} f(z)\overline{g(z)} + e^{-\pi H(z,z)} \frac{\partial f(z)}{\partial \overline{z}_j}\overline{g(z)} + e^{-\pi H(z,z)} f(z)\overline{\left(\frac{\partial g(z)}{\partial z_j}\right)} =$$
$$= \left(\frac{\partial}{\partial \overline{z}_j} f, g\right)_{pt} - \left(f, \frac{\partial}{\partial \overline{z}_j}^* g\right)_{pt}$$

Now, if we take complex conjugates on the identity

$$\frac{\partial}{\partial \overline{z}_j}(f,g)_{pt} = \left(\frac{\partial}{\partial \overline{z}_j} f, g\right)_{pt} - \left(f, \frac{\partial}{\partial \overline{z}_j}^* g\right)_{pt}$$

We get

$$\frac{\partial}{\partial z_j}(g,f)_{pt} = \overline{\frac{\partial}{\partial \overline{z}_j}(f,g)_{pt}} = \left(g, \frac{\partial}{\partial \overline{z}_j} f\right)_{pt} - \left(\frac{\partial}{\partial \overline{z}_j}^* g, f\right)_{pt}$$

Which shows the remaining identity, AND ALSO THAT THE FORMULA AT THE TOP OF PAGE 24 IS WRONG.

3. He says just apply the formula, but at this point it seems natural to be wary. Take a function f:

$$\frac{\partial}{\partial \overline{z}_j}\left(-\frac{\partial}{\partial z_j} + \pi h^j \overline{z}_j\right) f(z) - \left(-\frac{\partial}{\partial z_j} + \pi h^j \overline{z}_j\right)\frac{\partial}{\partial \overline{z}_j} f(z) = \frac{\partial}{\partial \overline{z}_j}(\pi h^j \overline{z}^j f(z)) - \pi h^j \overline{z}^j \frac{\partial}{\partial \overline{z}_j} f(z) =$$
$$= \left(\frac{\partial \pi h^j \overline{z}^j}{\partial \overline{z}_j}\right) f(z) = \pi h^j f(z)$$

□

The inner product on A^n is diagonal

Computation (Lemma 3.7 c)). *Given that A^n is a big direct sum of copies of A^0 (where the basis is given by $d\overline{z}_I$, for $I \in \mathrm{Ind}$), the first two statements in A^0 are straightforward. For the third one, let us do the computation.*

Proof.

$$\overline{\partial}^*\overline{\partial} f(z) d\overline{z}_I = \overline{\partial}^* \sum_{j \notin I} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_I =$$
$$= \sum_{j \notin I} \frac{\partial}{\partial \overline{z}_j}^* \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_I + \sum_{j \notin I}\sum_{i_k \in I} (-1)^k \frac{\partial}{\partial \overline{z}_k}^* \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_{I-\{i_k\}}$$

$$\overline{\partial}\overline{\partial}^* f(z) d\overline{z}_I = \overline{\partial} \sum_{i_k \in I} (-1)^{k+1} \frac{\partial}{\partial \overline{z}_{i_k}}^* f d\overline{z}_{I-\{i_k\}} =$$
$$= \sum_{i_k \in I} \frac{\partial}{\partial \overline{z}_{i_k}} \frac{\partial}{\partial \overline{z}_{i_k}}^* f d\overline{z}_I + \sum_{j \notin I}\sum_{k \in I} (-1)^{k+1} \frac{\partial}{\partial \overline{z}_j} \frac{\partial}{\partial \overline{z}_{i_k}}^* f d\overline{z}_{I-\{i_k\}}$$

Using the fact that $\frac{\partial}{\partial \overline{z}_j}$ and $\frac{\partial}{\partial \overline{z}_{j'}}^*$ commute if $j \neq j'$, we have that indeed

$$(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) f(z) d\overline{z}_I = \sum_{i \in I} \frac{\partial}{\partial \overline{z}_i} \frac{\partial}{\partial \overline{z}_i}^* f d\overline{z}_I + \sum_{i \notin I} \frac{\partial}{\partial \overline{z}_i}^* \frac{\partial}{\partial \overline{z}_i} f d\overline{z}_I$$

□

Now, as we said at the start, the cohomology of the complex is going to be isomorphic to the kernel of $\Delta = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$.

The formula for the commutator of $\frac{\partial}{\partial \overline{z_j}}^*$ and $\frac{\partial}{\partial \overline{z_j}}$ says that this kernel equals

$$H_I = \left\{ f : \sum_{1\le j\le g} \frac{\partial}{\partial \overline{z_j}}^* \frac{\partial}{\partial \overline{z_j}} f = -\pi \sum_{j\in I} h^j f \right\} d\overline{z}_I$$

NOTICE THE SIGN ERROR AT THE TOP OF PAGE 25 Comments:

1. If the line bundle is the trivial bundle, then the differential equations say that all the functions must be harmonic (i.e. constant since X is compact), and we can easily see that the cohomology of $\mathcal{O}$ is an exterior algebra in g generators.
2. If we add to the set I some indices for which $h^i = 0$, the differential equation doesn't change at all!

Lemma (Lemma 3.8)**.** *If $H_I \neq 0$, then*

1. $I \subseteq \{j : h^j \le 0\}$
2. $\{j : h^j < 0\} \subseteq I$

Proof. 1. Let's prove the contrapositive, assume for $i \in I$, $h^j > 0$. Then let $f \in H_I$, so $\Delta f = 0$. Then (NOTICE THE TYPO IN THE FORMULA IN THE MIDDLE OF PAGE 25)

$$(\Delta f, f) = \sum_{1\le j\le n} \left(\frac{\partial}{\partial \overline{z_j}}^* \frac{\partial}{\partial \overline{z_j}} f, f \right) + \pi \sum_{j\in I} h^j (f, f)$$

Notice all the positive numbers in this equation! We must have that $\sum_I h^j < 0$ for this equation to hold with nonzero f. However, if $h^i > 0$, we can make it arbitrarily big by taking a diagonal change of coordinates.

2. Serre duality in this case says that the following is a perfect pairing:

$$\begin{array}{rcl} H_I(\alpha, H) \times H_{I^c}(\alpha^{-1}, -H) & \longrightarrow & H^g(X, \mathcal{O}) \cong \mathbb{C} \cdot d\overline{z}_{[n]} \\ (f d\overline{z}_I, g d\overline{z}_{I^c}) & \longmapsto & fg d\overline{z}_{[n]} \end{array}$$

Provided that this is well-defined, this is obvious, since multiplying two functions cannot give 0 as an answer! (remember these functions are harmonic).

The duality yields the second result in this case.

□

3 3.5: Profit

Computation. *The equations for the trivial line bundle easily imply that all coefficients are constant. Therefore, the cohomology of $\mathcal{O}$ is an exterior algebra in g generators.*

Let us fix notation: as before, we have $X = V/L$ with some Appel-Humbert data (α, H), H is diagonal.

Let N be the set of indices for which $h^j < 0$ and Z be the set such that $h^j = 0$.

Let $K = \ker H/(\ker H \cap L)$ be the biggest (connected) subtorus on which H is trivial, so the line bundle restricts to a topologically trivial line bundle (the work to show this is nice was done by Soumya in Corollary 1.9).

Let $\mathcal{L}$ be the line bundle corresponding to α, H.

Theorem (Theorem 3.9)**.** *1. $H^i(X, L) = 0$ unless $\#N \le i\#N + \#Z$.*

2. For any i

$$H^{\#N+i}(X, \mathcal{L}) \cong H^{\#N}(X, \mathcal{L}) \otimes H^i(K, \mathcal{O})$$

(But both sides are 0 unless $0 \le i \le \#Z = \dim K$)

3. *If* $\alpha|_K = 1$*, then*

$$\dim H^{\#n}(X,\mathcal{L}) = \sqrt{\det{}_{L/\ker H} \Im H}$$

Otherwise, the dimension is 0.

In Theorem 3.9, notice the stupid typo in part c), where it should say "im" instead of "dim".

Proof. 1. This is clear from the work we've done.

2. Let us name the coordinates like Kempf does: Let $z_1, \dots, \overline{z}_k$ be the coordinates where the hermitian form vanishes, and $z_{k+1}, \dots, z_{k+n}$ be the coordinates where it is negative definite. Then for any $J \subseteq \{1, \dots, k\}$, we take the map

$$f d\overline{z}_{k+1} \wedge \dots \wedge d\overline{z}_{k+n} \longmapsto f d\overline{z}_J \wedge d\overline{z}_{k+1} \wedge \dots \wedge d\overline{z}_{k+n}$$

This map is an isomorphism: the differential equations on both sides agree.

We can think of this as looking at the projection $\pi : X \to X/K$, and then $\mathcal{L} = \pi^* \widetilde{\mathcal{L}}$ (provided $\alpha|_{\ker H} = 1$, which is the only interesting case). Then

$$H^\bullet(X, \pi^* \widetilde{\mathcal{L}}) = H^\bullet(X/K, R\pi_*(\pi^* \widetilde{\mathcal{L}})) = H^\bullet(X/K, \widetilde{\mathcal{L}} \otimes R\pi_*(\mathcal{O})) = H^n(X/K, \widetilde{\mathcal{L}}) \otimes H^\bullet(K, \mathcal{O})$$

3. We do some magic which is unexplicable to me, other than the fact that it works wonderfully. We take a new coplex structure on V where the old coordinates are holomorphic, except for $z_{k+1}, \dots, z_{k+n}$, where we make them antiholomorphic instead. So the holomorphic chart is now $z_1, \dots, z_k, \overline{z}_{k+1}, \dots, \overline{z}_{k+n}, z_{k+n+1}, \dots, z_n$. We take the Hermitian form on the new vector space $\widehat{V}$ given by

$$\widehat{H}(z, w) = \sum_{j \in N} -h^j \overline{z}_j w_j + \sum_{j \notin N} h^j z_j \overline{w}_j$$

Notice that they have the same imaginary part, but $\widehat{H}$ is semipositive definite. The magic is that

$$H^{\#N}(X, \mathcal{L}) \cong H^0(\widehat{V}/L, \mathcal{L}(\alpha, \widehat{H}))$$

Now the theorem is exactly the computation of H^0 that Michael did in section 2.

By the arguments we did at the start, a form $f d\overline{z}_N$ being in $H_N(X, \mathcal{L})$, i.e. being harmonic, is the same as having $\overline{\partial} f = \overline{\partial}^* f = 0$. The isomorphism is

$$f d\overline{z}_N \longmapsto g = e^{-\pi \sum_{j \in N} h^j z_j \overline{z}_j} f$$

We can show easily that g is holomorphic. Let $j \notin N$. Then

$$\frac{\partial}{\partial \overline{z_j}} g = e^{-\pi \sum_{j \in N} h^j z_j \overline{z}_j} \frac{\partial}{\partial \overline{z_j}} f = 0$$

If $j \in N$, then we must take the derivate with respect to z_j:

$$\frac{\partial}{\partial z_j} g = \frac{\partial}{\partial z_j} \left(e^{-\pi \sum_{j \in N} h^j z_j \overline{z}_j} f \right) = -h^j \overline{z}_j e^{-\pi \sum_{j \in N} h^j z_j \overline{z}_j} f + e^{-\pi \sum_{j \in N} h^j z_j \overline{z}_j} \frac{\partial f}{\partial z_j} = e^{-\pi \sum_{j \in N} h^j z_j \overline{z}_j} \frac{\partial}{\partial \overline{z_j}}^* f = 0$$

We also have to check that g has the correct equivariance, i.e.

$$\begin{aligned} g(z + l) &= e^{-\pi \sum_{j \in N} h^j (z_j + l_j)(\overline{z}_j + \overline{l}_j)} f(z + l) = \\ &= e^{-\pi \sum_{j \in N} h^j (z_j + l_j)(\overline{z}_j + \overline{l}_j)} \alpha_l e^{\pi H(z,l) + \frac{\pi}{2} H(l,l)} f(z) = \\ &= \alpha_l e^{\pi \sum_{j \notin N} h^j z_j \overline{l}_j - \pi \sum_{j \in N} h^j \overline{z}_j l_j - \frac{\pi}{2} \widehat{H}(l,l)} e^{-\pi \sum_{j \in N} h^j z_j \overline{z_j}} f(z) = \\ &= \alpha_l e^{\pi \widehat{H}(z,l) - \frac{\pi}{2} \widehat{H}(l,l)} g(z) \end{aligned}$$

Alternatively, let us write $H = H^+ - H^-$, where both H^+ and H^- are semipositive definite and they have the smallest minimal rank (i.e. split the positive and negative coefficients). Then $\widehat{H} = H^+\overline{H^-}$. We have that $H = \widehat{H} + 2\Re H^-$, and therefore

$$\begin{aligned} g(z+l) &= e^{-\pi H^-(z+l,z+l)} f(z+l) = \\ &= e^{-\pi H^-(z+l,z+l)} \alpha_l e^{\pi H(z,l)+\frac{\pi}{2}H(l,l)} f(z) = \\ &= \alpha_l e^{-2\pi\Re H^-(z,l)-\pi H^-(l,l)+\pi\widehat{H}(z,l)+2\pi\Re H^-(z,l)+\frac{\pi}{2}\widehat{H}(l,l)+\pi\Re H^-(l,l)} e^{-\pi H^-(z,z)} f(z) = \\ &= \alpha_l e^{\pi\widehat{H}(z,l)-\frac{\pi}{2}\widehat{H}(l,l)} g(z) \end{aligned}$$

□

4 3.6 Examples

Let $X = V/L$ as usual.

Theorem (Theorem 3.10)**.**

$$\chi(\mathcal{L}) = \frac{c_1(\mathcal{L})^g}{g!}$$

Proof. Let $\mathcal{L} = \mathcal{L}(\alpha, H)$. Recall that $c_1(\mathcal{L}) = \Im H$, where we can view $\Im H$ as a symplectic form on $L = H_1(X, \mathbb{Z})$, i.e. an element of $H^2(X, \mathbb{Z})$. Then c_1^g is the determinant of E, i.e. as a matrix it's $\det E^t E$ (when we use as a basis a basis for L).

If $c_1^g = 0$ it means that H is degenerate, and in this case by the previous theorem we have an exterior algebra showing up somewhere, which means that $\chi = 0$.

If $c_1^g \neq 0$ (i.e. E is full rank), then all the cohomologies vanish except for $H^{\# N}$, where N is the subset of the basis where H is negative definite. Then the determinant equals $H^{\# N}$ by the theorem, and the sign is apparently correct (I haven't checked this). □

We have a nice corollary when $\mathcal{L}$ is ample (i.e. when H is positive definite).

Let us see what happens for the Poincaré bundle.

Computation. *Let $\mathcal{P}$ be the Poincaré bundle. Then its only nonzero cohomology group is H^g, which is one dimensional.*

Proof. Recall that $X^\vee = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})/L^\vee$. In the book $\mathrm{Hom}(V, \mathbb{R})$ has the complex structure given by $i \cdot phi = -\varphi \cdot i$. However, given $\varphi \in \mathrm{Hom}(V, \mathbb{R})$ we can get a $\widetilde{\varphi} \in \mathrm{Hom}_{\mathbb{C}}(\overline{V}, \mathbb{C}) = \overline{V^\vee}$ by setting $\widetilde{\varphi}(v) = \varphi(v) + i\varphi(iv)$ (which would make it antiholomorphic), and this is a bijection whose inverse is $\varphi = \Re\widetilde{\varphi}$. Then $L^\vee$ is the subset of $\overline{V^\vee}$ where $\varphi(L) \subseteq \mathbb{Z} + i\mathbb{R}$.

The canonical Hermitian form on $V \times \overline{V^\vee}$ is $(z, \varphi) \mapsto i\overline{\varphi(z)}$, which has matrix, if we pick a basis on V and its dual on $\overline{V^\vee}$,

$$\left(\begin{array}{c|c} 0 & -iI \\ \hline iI & 0 \end{array}\right)$$

Which has $-g$ negative eigenvalues and g positive ones. This implies by the Theorem that $\mathcal{P}$'s only nonzero cohomology is at degree g.

If we now write the lattice L as the standard basis e_i union $v_1, \dots v_g$ (some other g vectors whose imaginary part forms an invertible matrix), then these form an $\mathbb{R}$-basis of V, so we can take the dual basis $\{e_i^\vee, v_i^\vee\}$, and then $\{\widetilde{e}_i^\vee, \widetilde{v}_i^\vee\}$ is a basis for $L^\vee$.

In this basis, $\Im H = (l, l') \mapsto l'(l)$ has matrix

$$\left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right)$$

Which has determinant 1, and this completes the calculation. □

4.1 Pushforwards of families of line bundles

When dealing the pushforwards, we want to answer the question about when a (higher) pushforward is locally free, and that answer is given often by Grauert's theorem. As stated in Ravi, this is

Theorem (Grauert's theorem)**.** *If $\pi : X \to Y$ is proper, Y is reduced and locally Noetherian, F is a coherent sheaf on X flat over Y, and $\dim H^p(X_y, F|_{X_y})$ is a locally constant function of $y \in Y$, then $R^p\pi_*F$ is locally free, and the following natural map is an isomorphism for any $y \in Y$:*

$$\varphi_y^p : (R^p\pi_*F)|_y \longrightarrow H^p(X_y, F|_{X_y})$$

Suppose $\pi : X \to S$ is a (proper, smooth hence flat) family of abelian varieties over S, and there's a line bundle X which is ample on every fiber. Then $H^0(X_s, \mathcal{L}|_{X_s})$ is locally constant, since $\chi(\mathcal{L}_s)$ is locally constant and there's no higher cohomologies. Therefore, using Grauert's theorem, $\pi_*\mathcal{L}$ is a vector bundle of rank equal to $\dim \Gamma(X_s, \mathcal{L}|_{X_s})$. Grauert's theorem applied to $p > 0$ says that higher pushforwards vanish.

Theorem (Theorem 3.15)**.** *Take $\mathcal{P}$ on $X \times X^\vee$, and let π be the second projection. Then*

$$R^g\pi_*\mathcal{P} \cong \mathcal{O}_0$$

And all the other pushforwards vanish.

Proof. We apply Grauert's theorem to all the nonzero fibers, which says that (since cohomology of bundles in Pic^0 is 0 unless the line bundle is trivial), all the pushforwards are 0.

Now we use the composition $R\Gamma(X \times X^\vee, \bullet) = R\Gamma(X^\vee, \bullet) \circ R\pi_*$, which applied to $\mathcal{P}$ yields

$$\mathbb{C}[-g] = R\Gamma(X \times X^\vee, \mathcal{P}) = R\Gamma(X^\vee, R\pi_*\mathcal{P}) \overset{\mathrm{supp} R\pi_*\mathcal{P}=\{0\}}{\cong} H^0(X^\vee, R\pi_*\mathcal{P})$$

□