1. Find the nonnegative integer $a < 28$ which is represented by the following pairs

   $(a)$ $(0, 0)$ $(b)$ $(1, 1)$

   $(c)$ $(2, 1)$ $(d)$ $(3, 5)$

   where each pair $(\kappa, \ell)$ represents the system of congruences

$$\left.\begin{array}{l} a \equiv \kappa \mod 4 \\ a \equiv \ell \mod 7 \end{array}\right\}.$$

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2. Using Fermat's little theorem show that if $n$ is a positive integer, $n^7 \equiv n \mod 42$.

   *Note:* Fermat's little theorem will be stated and proved next Tuesday in class. It states that $a^{p-1} \equiv 1 \mod p$ for any prime $p$ and any integer $a$ so that $p \nmid a$. Equivalently $a^p \equiv a \mod p$ for any integer $a$.

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3. Let $m_1, m_2 > 1$. Show that the system of linear congruences

$$\left.\begin{array}{l} x \equiv a \mod m_1 \\ x \equiv b \mod m_2 \end{array}\right\}$$

   has solutions **for any** integers $a$ and $b$ if, and only if, $m_1$ and $m_2$ are relatively prime.

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4. Let $\varphi(m) = \left\{1 \leq k < m \;/\; \gcd(k, m) = 1\right\}$ be Euler's function. Show that:

   $(a)$ For any prime $p$ and any integer $\kappa \geq 1$, $\varphi\left(p^\kappa\right) = p^{\kappa-1}(p-1)$.

   $(b)$ Use the multiplicative property of $\varphi$ to prove that if $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$ is the prime factorization of $m$, then

$$\varphi(m) = m\left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_k}\right).$$

   $(c)$ Use $(b)$ to show that, in particular, for any integer $\kappa \geq 1$, $\varphi\left(m^\kappa\right) = m^{\kappa-1}\varphi(m)$.

   *Note:* Recall that $\varphi$ being multiplicative means that $\varphi(n \cdot m) = \varphi(n) \cdot \varphi(m)$ if $m, n \geq 1$ are relatively prime.

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5. Let $p$ and $q$ be two different primes, put $m = pq$ and suppose that $r \equiv 1 \mod (p-1)$ and $r \equiv 1 \mod (q-1)$. Show that for any integer

$$a^r \equiv a \mod m.$$