1. Find the nonnegative integer $a < 28$ which is represented by the following pairs

   (a) $(0, 0)$ (b) $(1, 1)$

   (c) $(2, 1)$ (d) $(3, 5)$

   where each pair (κ, ℓ) represents the system of congruences

$$\left.\begin{aligned} a &\equiv \kappa \mod 4 \\ a &\equiv \ell \mod 7 \end{aligned}\right\}.$$

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2. Using Fermat's little theorem show that if n is a positive integer, $n^7 \equiv n \mod 42$.

   Note: Fermat's little theorem will be stated and proved next Tuesday in class. It states that $a^{p-1} \equiv 1 \mod p$ for any prime p and any integer a so that $p \nmid a$. Equivalently $a^p \equiv a \mod p$ for any integer a.

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3. Let $m_1, m_2 > 1$. Show that the system of linear congruences

$$\left.\begin{aligned} x &\equiv a \mod m_1 \\ x &\equiv b \mod m_2 \end{aligned}\right\}$$

   has solutions **for any** integers a and b if, and only if, m_1 and m_2 are relatively prime.

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4. Let $\varphi(m) = \#\left\{1 \leq k < m \;/\; \gcd(k, m) = 1\right\}$ be Euler's function. Show that:

   (a) For any prime p and any integer $\kappa \geq 1$, $\varphi\left(p^\kappa\right) = p^{\kappa-1}(p-1)$.

   (b) Use the multiplicative property of φ to prove that if $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$ is the prime factorization of m, then

$$\varphi(m) = m\left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_k}\right).$$

   (c) Use (b) to show that, in particular, for any integer $\kappa \geq 1$, $\varphi\left(m^\kappa\right) = m^{\kappa-1}\varphi(m)$.

   Note: Recall that φ being multiplicative means that $\varphi(n \cdot m) = \varphi(n) \cdot \varphi(m)$ if $m, n \geq 1$ are relatively prime.

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5. Let p and q be two different primes, put $m = pq$ and suppose that $r \equiv 1 \mod (p-1)$ and $r \equiv 1 \mod (q-1)$. Show that for any integer a,

$$a^r \equiv a \mod m.$$